Many-body chaos and semiclassical limit of a measurement-induced phase transition

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S. Ruidas & SB, **SciPost Phys. 11**, 087 (2021).

SB & E. Altman, **PRB 95**, 134302 (2017).

S. Bera, V. Lokesh K Y & SB, **PRL 128**, 115302 (2022)

A. Haldar, SB, V. B. Shenoy, **PRB (Rapid) 97**, 241106 (2018).

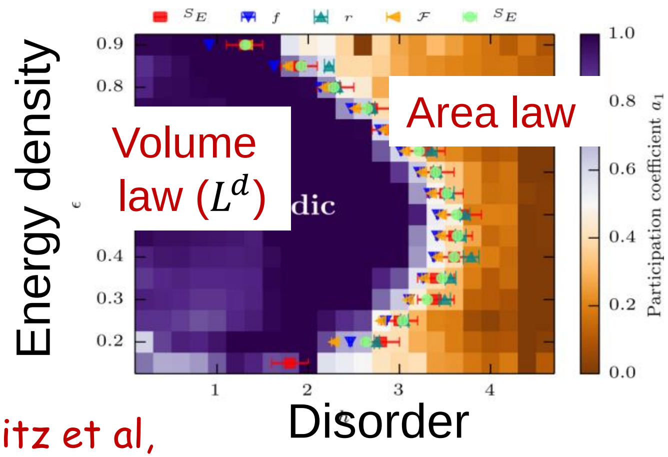
S. Ruidas & SB, arXiv:2210.03760

Sibaram Ruidas
(ICTS)

“Chaotic to non-chaotic phase transitions” in quantum many-body systems

Many-body Localization (MBL)

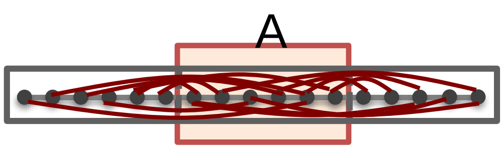
Basko et al. (2005); Gornyi et al. (2005)



Luitz et al, PRB (2015)

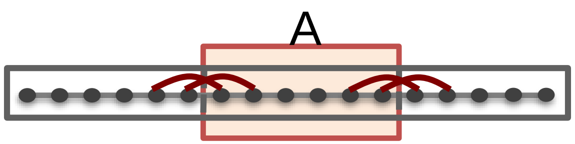
Volume law

$S_A \sim L^d$



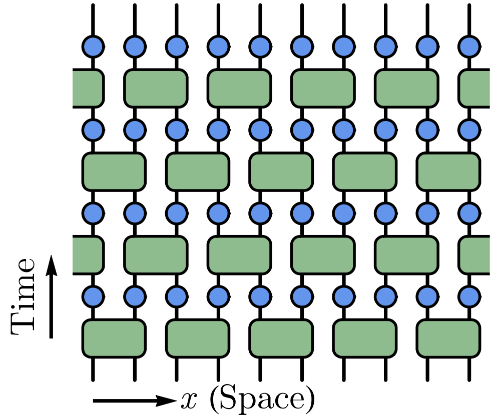
Area law

$S_A \sim L^{d-1}$



→ New paradigm for phase transitions   Entanglement transitions

More tractable models – Random quantum circuits with non-unitary evolution



Unitary evolution

+

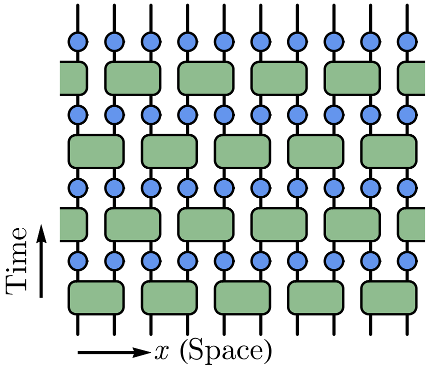
measurements

local projective or weak measurements

p fraction of sites are measured in each interval

Tune measurement rate p, measurement strength, unitary strength, etc., ..

Measurement-induced phase transition (MIPT)



Skinner et al. (2019), Bao et al. (2019);
Li et al. (2019), Jian et al. (2019),
Gullans et al. (2020), Nahum et al. (2021)
Sang et al. (2021), ...

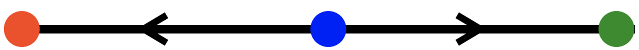
Volume law
$S_A \sim L^d$

Area law
$S_A \sim L^{d-1}$

Entangling phase

Critical dynamics

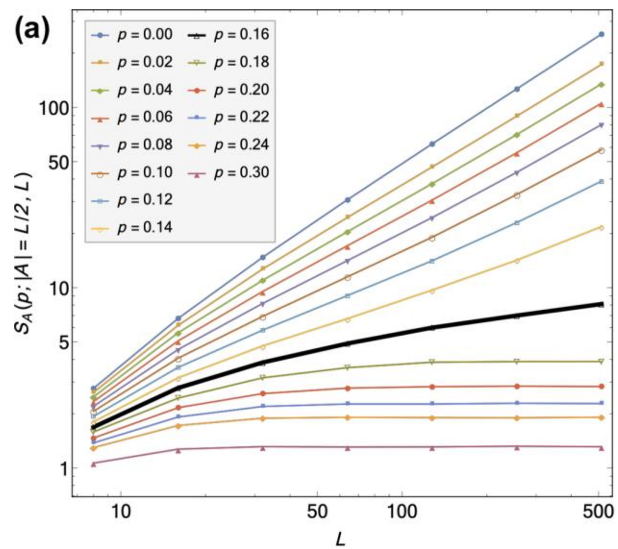
Disentangling phase



$p < p_c$

p_c

$p > p_c$

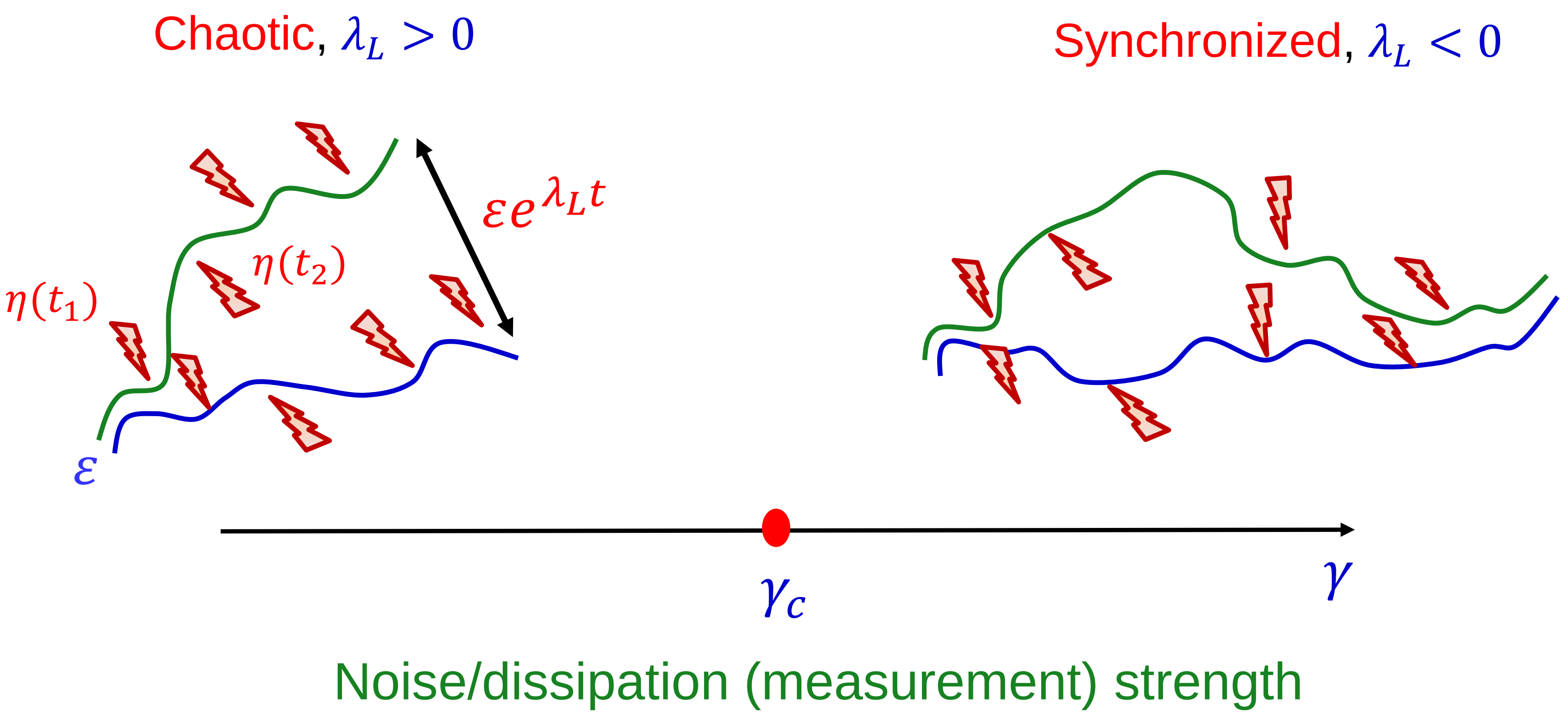


Transition in steady state from volume-law to area-law entanglement

- Continuous weak measurements Szyniszewski et al. (2019, 2020), ..
- Non-interacting fermions Cao et al. (2019), Chen et al. (2020), Alberton et al. (2021),..
- Interacting bosons Tang et al. (2020), Fuji et al. (2020), Goto et al. (2020), ..
- Luttinger liquids Garratt et al. (2020)

This talk

A possible connection between measurement-induced phase transition (MIPT) with chaotic-to-non-chaotic transition (stochastic synchronization) in classical system



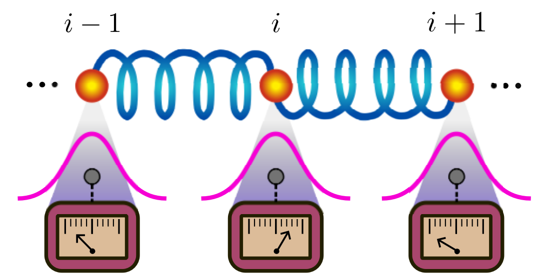
Outline of the talk

- Overview of many-body chaos in classical and quantum systems

- Semiclassical limit of a model of continuous weak measurements

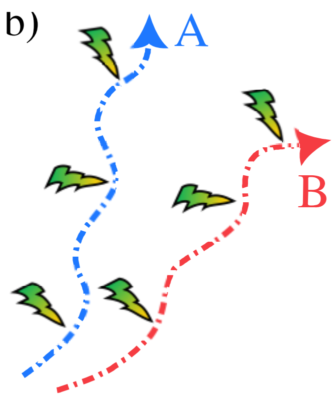
  ⇒ Stochastic Langevin equation

  noise/dissipation ∝ "measurement strength"

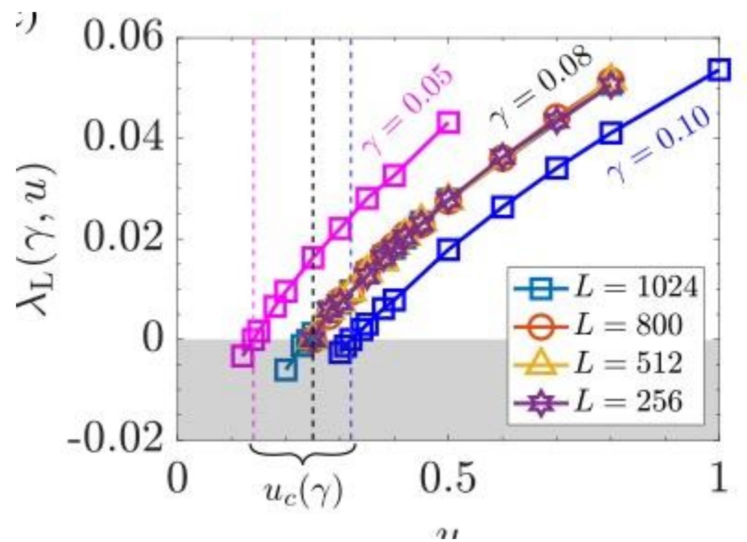


- Noise/measurement induced chaotic to non-chaotic transition

  Stochastic synchronization transition



Classical OTOC



Non-integrable and integrable (Toda) interacting oscillator chains

Classical Chaos

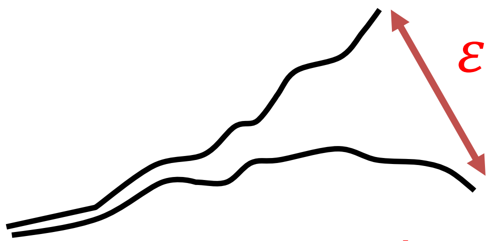
"When present determines the future, but approximate present does not approximately determine the future."

Edward Lorenz

Single-particle chaos

Sensitivity to initial condition ⇒

ε



$$\varepsilon e^{\lambda_L t} \lesssim \mathcal{O}(1)$$

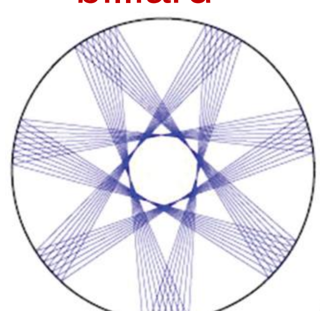
Lyapunov regime

$$\lambda_L^{-1} < t < t^* \sim \lambda_L^{-1} \log\left(\frac{1}{\varepsilon}\right)$$

$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda_L t}$$

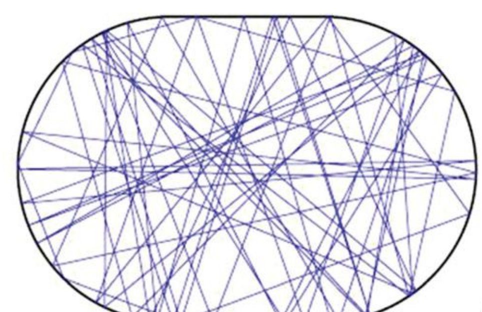
λ_L, Lyapunov exponent

Non chaotic billiard



(a)

Chaotic billiard



Classical many-body chaos

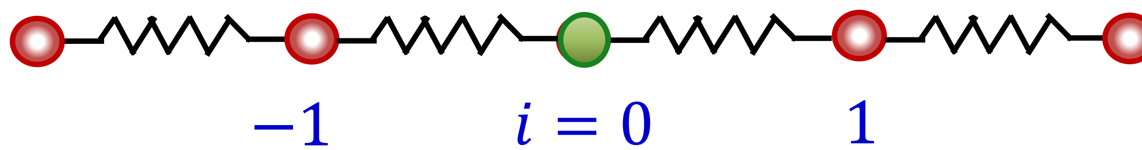
Example1: Anharmonic coupled oscillator chain

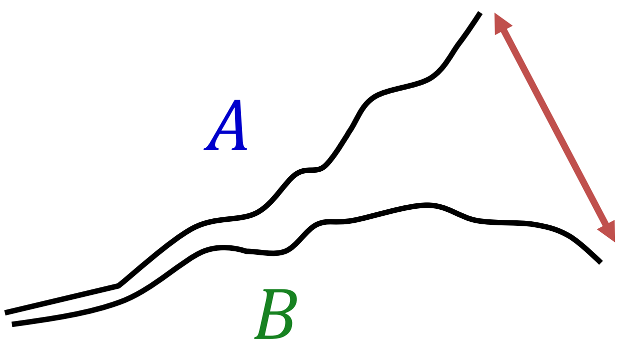
Newtonian dynamics

$$\ddot{x}_i = -\frac{\partial V(\{x_i\})}{\partial x_i}$$

$i = 1, \ldots, N$

$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1} - x_i)^2 + \frac{u}{4}(x_{i+1} - x_i)^4\right]$$





$$x_i^A(0) - x_i^B(0) = \varepsilon\delta_{i,0}$$

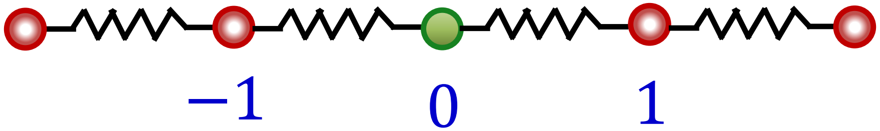
Two trajectories with slightly different initial conditions at $i = 0$ at time $t = 0$

Classical OTOC or decorrelation function

$$D(i,t) = \left\langle \left(x_i^A(t) - x_i^B(t)\right)^2 \right\rangle_T$$

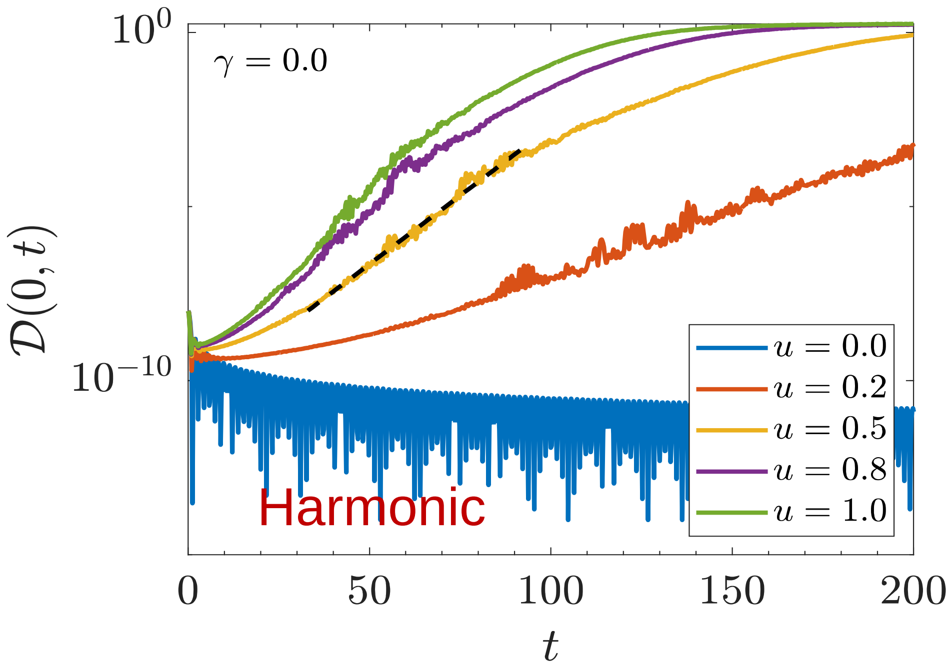
$\langle \cdots \rangle$

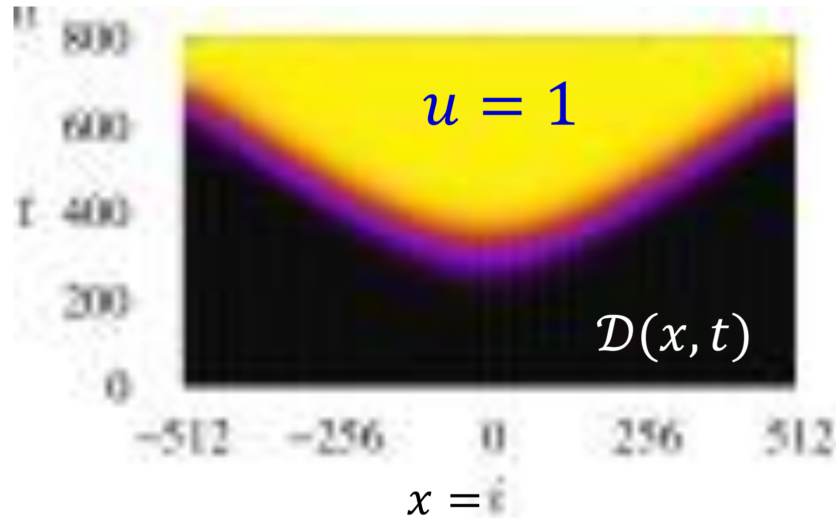
Thermal initial condition at temperature T



$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1}-x_i)^2 + \frac{u}{4}(x_{i+1}-x_i)^4\right]$$

Ballistic spread of chaos in a diffusive system





Light cone spread with butterfly velocity $v_B \neq 0$

$$D(x,t) \sim e^{\lambda_L t\left(1-\left(\frac{x}{v_B t}\right)^2\right)}$$

- $\lambda_L = 0$ in the non-interacting (harmonic) case $u = 0$.
- $\lambda_L > 0$ in the interacting case $u \neq 0$.

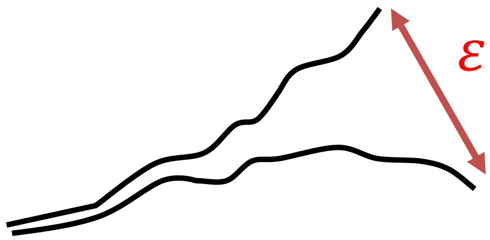
Two chaos parameters $\lambda_L,\ v_B$

Quantum Chaos

Classical single-particle chaos

$$\frac{\delta x(t)}{\delta x(0)} \sim e^{\lambda_L t}$$

λ_L, Lyapunov exponent



ε

$$\varepsilon e^{\lambda_L t} \lesssim \mathcal{O}(1)$$

Quantum chaos

Larkin & Ovchinikov (1969)

- Classical chaos

$$\frac{\partial x(t)}{\partial x(0)} = \{x(t), p(0)\} \Rightarrow [x(t), p(0)]/i\hbar$$

Poisson bracket

Out-of-time order commutator

$$\mathcal{D}(t) = -\langle [x(t), p(0)]^2 \rangle \sim \hbar^2 \, e^{2\lambda_L t}$$

Lyapunov regime

$$\lambda_L^{-1} < t < t^* \sim \lambda_L^{-1} \log\left(\frac{1}{\hbar}\right)$$

Small parameter

"ε" $\equiv \hbar$

Many-body quantum chaos: Scrambling of quantum information

Generalize to quantum chaotic (interacting) many-body systems

$$\mathcal{D}(t) = -\langle [A(t), B(0)]^2 \rangle \sim e^{\lambda_L t} \;\Rightarrow\; \text{Quantum Lyapunov exponent } \lambda_L$$

$$A(t) = e^{i\mathcal{H}t} A e^{-i\mathcal{H}t} = A + it[\mathcal{H}, A] - \frac{t^2}{2!}[\mathcal{H},[\mathcal{H},A]] - \frac{it^3}{3!}\big[\mathcal{H},[\mathcal{H},[\mathcal{H},A]]\big] + \cdots$$

Local operator grows in size encompassing the whole system Scrambling

Out-of-time-order correlator (OTOC)

$$F(t) = \langle A(t)B(0)A(t)B(0) \rangle$$
$$\sim \# - \mathcal{D}(t) \sim \# - \epsilon e^{\lambda_L t}$$

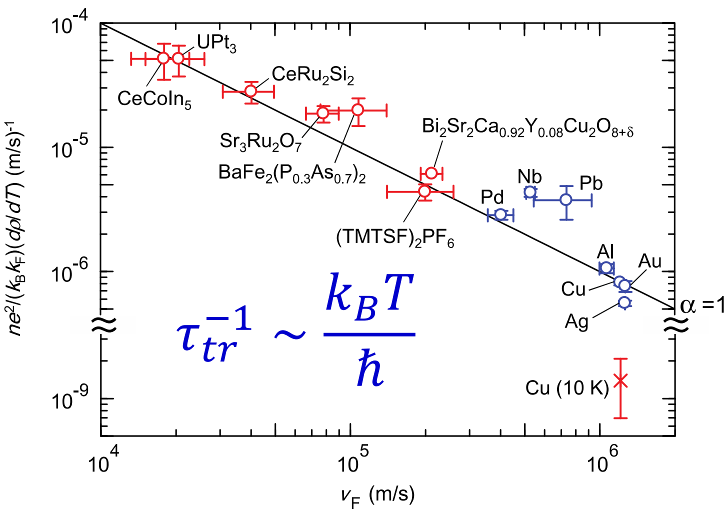
○ Remarkable upper bound for Lyapunov exponent

$$\lambda_L \le 2\pi k_B T/\hbar$$

Maldacena, Shenker & Stanford (2016)

○ Are there bounds on transport scattering rate τ_{tr}^{-1}?

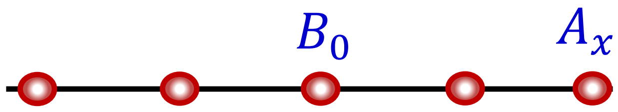
Bruin et al. Science (2013)



Butterfly velocity v_B

$$\mathcal{D}(x,t) = -\langle [A_x(t), B_0(0)]^2 \rangle \sim e^{\lambda_L\left(t - \frac{|x|}{v_B}\right)}$$

or $$\sim e^{\lambda_L t\left(1 - \left(\frac{x}{v_B t}\right)^2\right)}$$

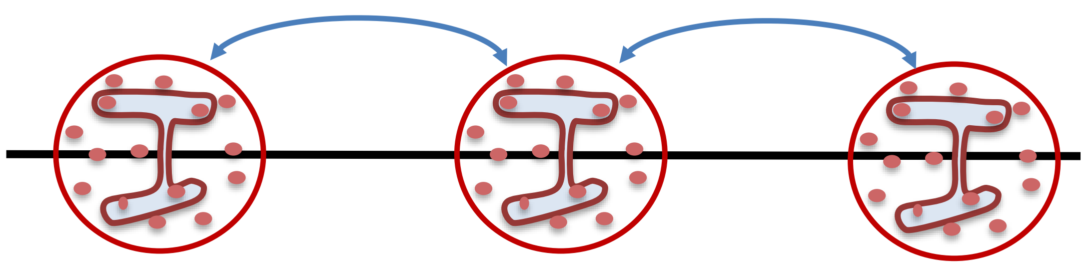


- In certain strongly correlated systems

  Diffusion coefficient $D \sim v_B^2/\lambda_L$

  Gu, Qi & Stanford (2017)

Relation between transport and chaos !!



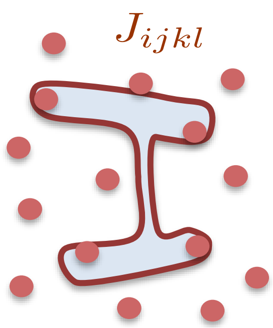
Lattice of SYK dots

Characterization of phases and phase transitions of quantum many-body systems in terms of chaos?

Chaos as an "Order Parameter"

Solvable models for chaos and chaotic transitions

Sachdev-Ye-Kitaev (SYK) model for a non-Fermi liquid: Maximal chaos



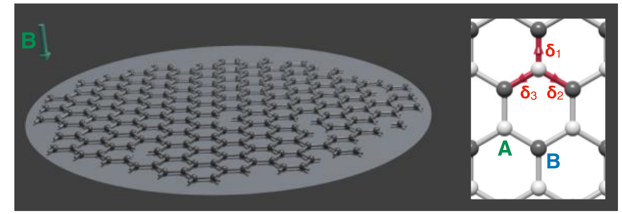
N sites

$$H_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l$$

Kitaev, KITP (2015),
Sachdev & Ye, PRL (1993)

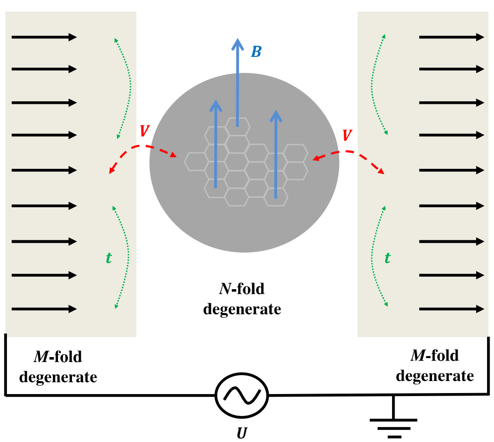
Lyapunov exponent, $\lambda_L = \dfrac{2\pi k_B T}{\hbar}$

Quantum Holography in a graphene flake

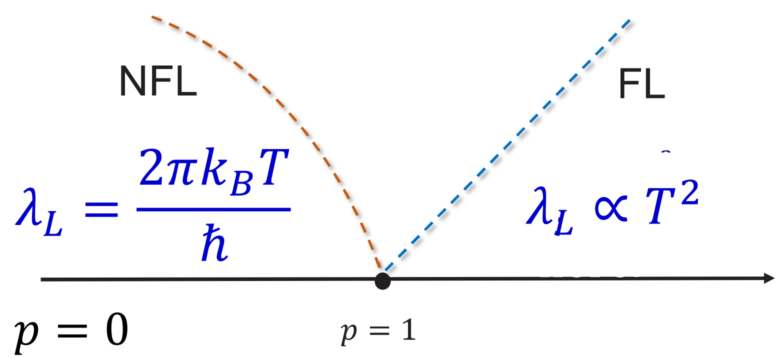


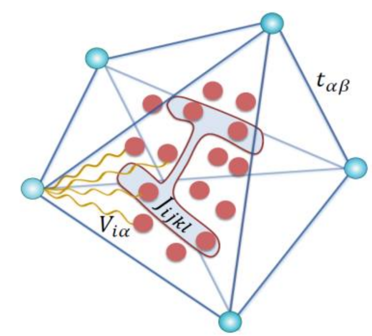
Chen et al., PRL (2018)

Can et al. PRB (2019)



Generalized SYK model for chaotic quantum phase transition (QPT)

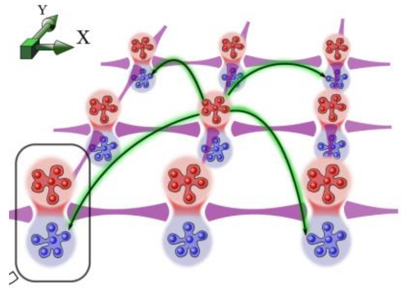




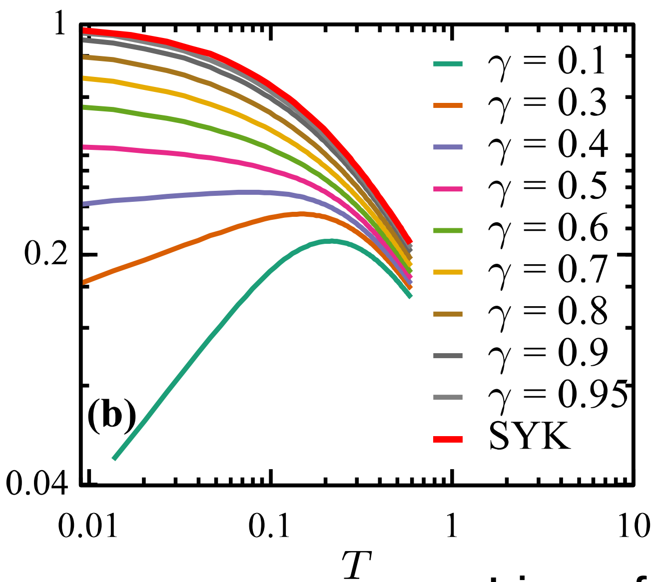
SB & E. Altman, PRB (2017)

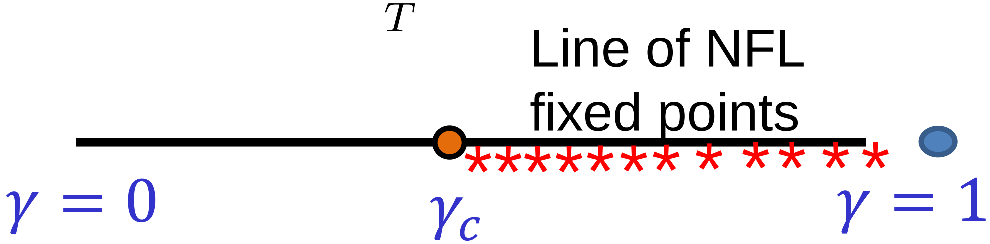
Other solvable chaotic transitions

Higher dimensional non Fermi liquids



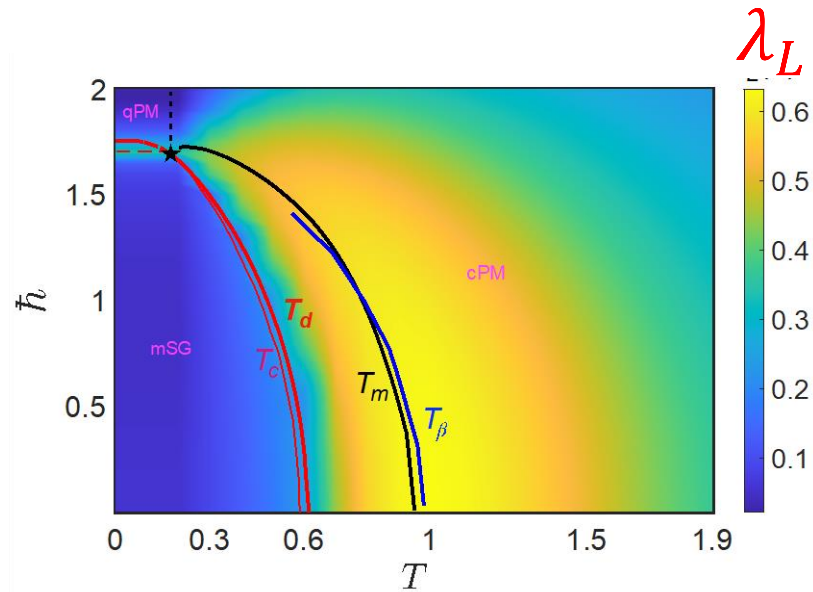
$\lambda_L = \alpha T$, $\alpha \leq 2\pi$





A. Haldar, SB, V. B. Shenoy, **PRB (Rapid) 97**, 241106 (2018).

Quantum p-spin glass



S. Bera, V. Lokesh K Y & SB, **PRL 128**, 115302 (2022)

A few comments

- Observability of Lyapunov growth

$$\mathcal{D}(t) \sim \varepsilon e^{\lambda_L t}$$

requires small 'semiclassical parameter' ε

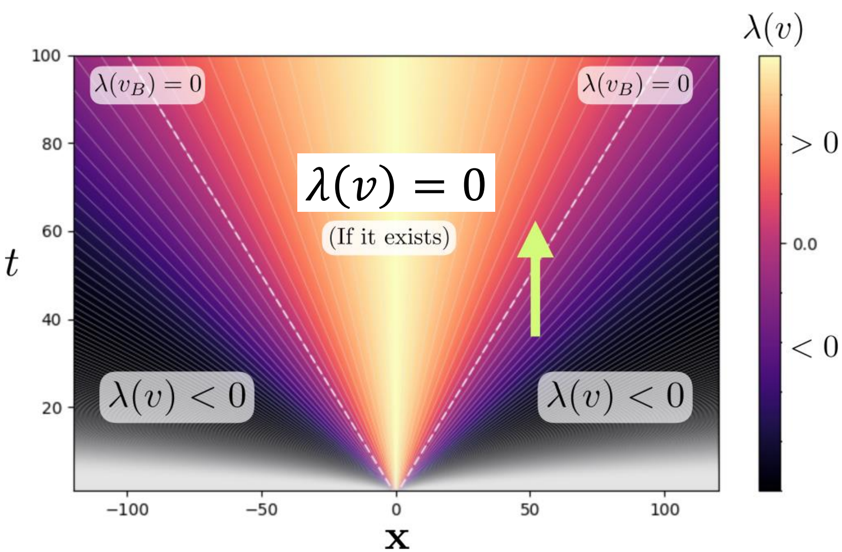
$$\lambda_L^{-1} < t < t^* \sim \lambda_L^{-1} \log\left(\frac{1}{\varepsilon}\right)$$

$\varepsilon \sim \hbar$ Semiclassical systems

$\sim \frac{1}{N}$ SYK-type models (large N models)

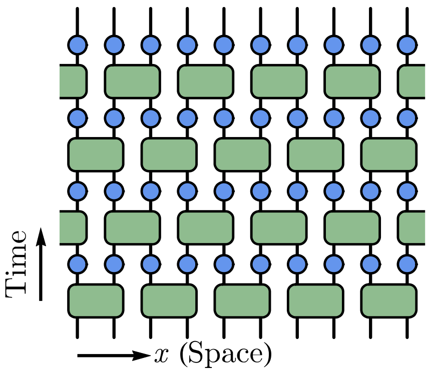
Local quantum systems with finite-dimensional local Hilbert space
(spin 1/2 spin chains, Hubbard model, ..)
No such small parameter
$\Rightarrow$ $\lambda_L > 0$ does not exist,
no exponential growth of OTOC

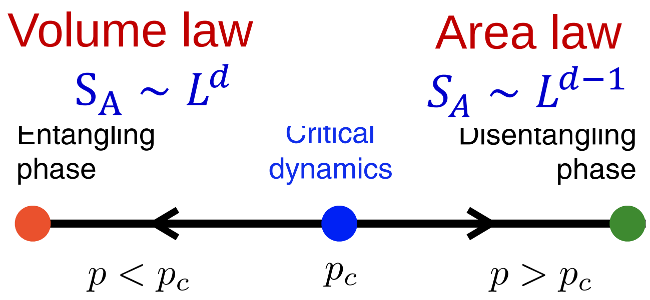
Velocity-dependent Lyapunov exponent

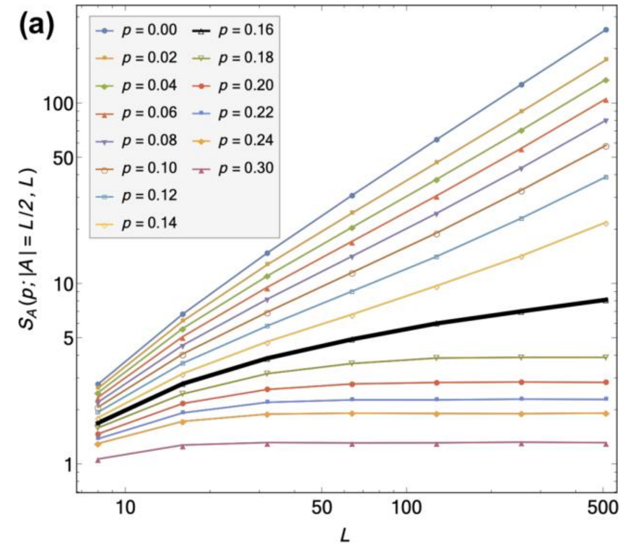


Khemani et al. PRB (2018)

Random matrix theory (RMT) or eigenstate thermalization hypothesis (ETH) is much better way to characterize chaos in usual quantum systems than OTOC







- Relation between chaos and entanglement?

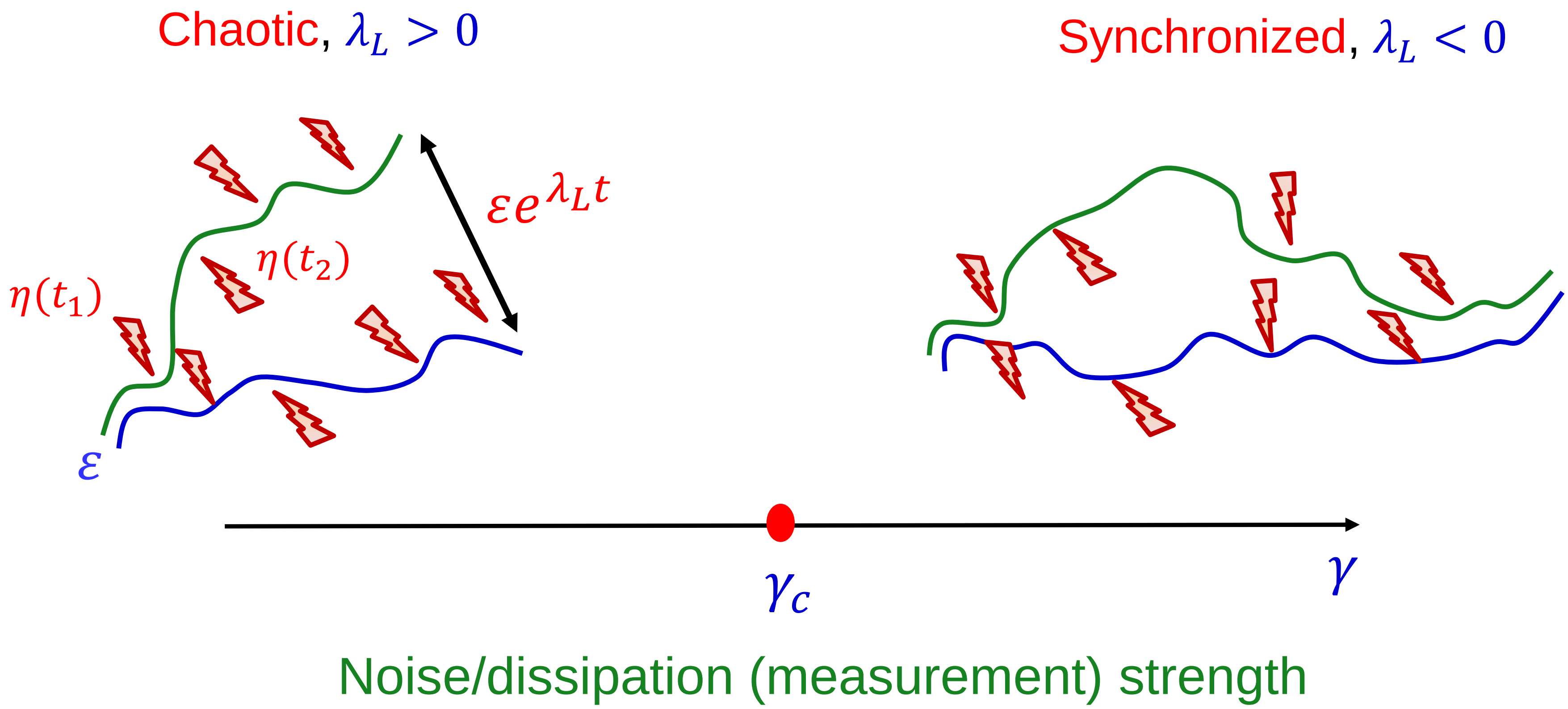
In certain types of OTOC
and non-equilibrium evolution

Fan et al. (2017);
Hosur et al. (2016);
Touil & Deffner(2020)

$$OTOC(t)\big|_{equil} \sim \exp\left[-S_A^{(2)}(t)\right]\big|_{non-equil}$$

$S_A^{(2)}$, Second Renyi entropy

A possible connection between measurement-induced phase transition (MIPT) with chaotic-to-non-chaotic transition (stochastic synchronization) in classical system

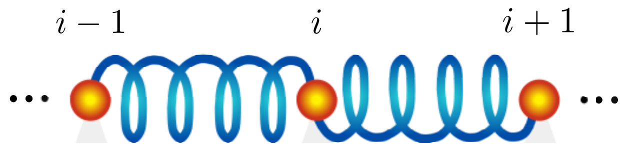


Quantum model of continuous weak measurements

Continuous (weak) position measurements of a free particle

Caves and Milburn, Phys. Rev. A 36 (1987)

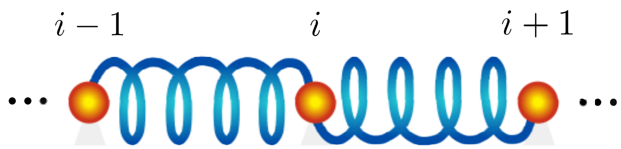
Generalize to interacting system of chains of anharmonic oscillators



$$H_s = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + V(\{\hat{x}_i\})$$

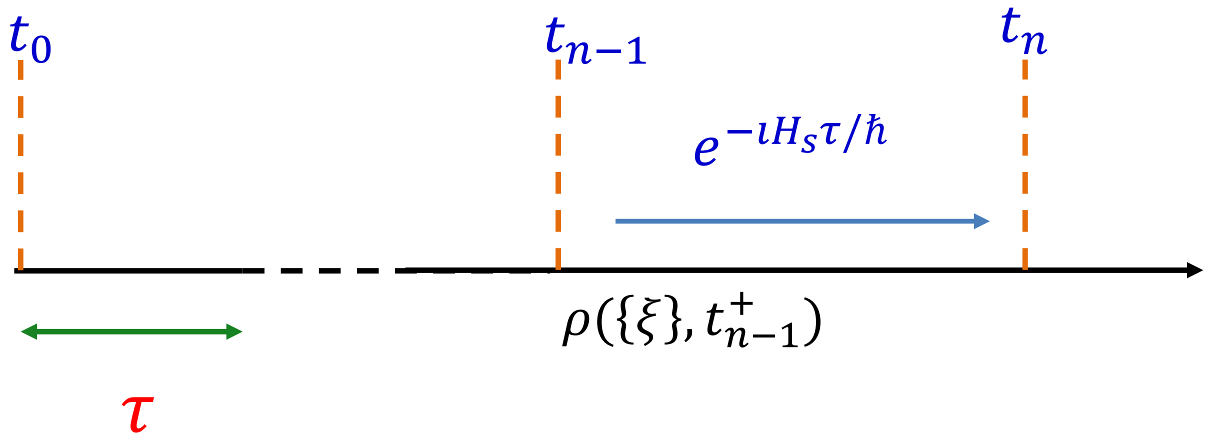
Quantum model of weak measurements

System under repeated weak measurements in intervals of τ



$t_n = n\tau$

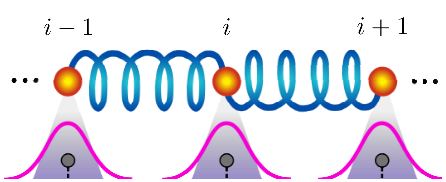
$$H(t) = H_s + \sum_{i,n} \delta(t - t_n)\, \hat{x}_i\, \hat{p}_{in}$$



Projective measurements of the positions of the meters

$$P_\xi = |\{\xi_{in}\}\rangle\langle\{\xi_{in}\}|$$

t_n^-

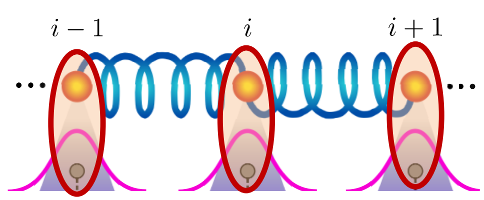


Meters

$\hat{x}_{in}, \hat{p}_{in}$

$$\psi(x_{in}) \sim \exp\left(-\frac{x_{in}^2}{2\sigma}\right)$$

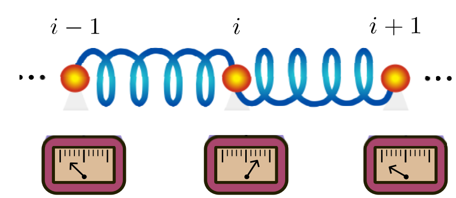
t_n



Apply

$$\sum_i \delta(t - t_n)\, \hat{x}_i\, \hat{p}_{in}$$

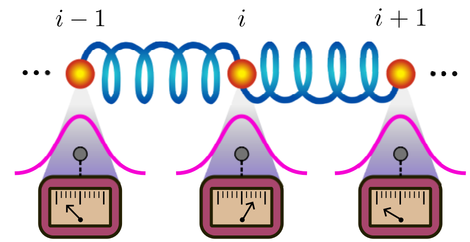
t_n^+



Readings

$\{\xi_{in}\}$

Evolution of density matrix



$$\rho(\{\xi\}_n, t_n^+) = M(\xi_n) e^{-\frac{\imath H_S \tau}{\hbar}} \rho(\{\xi\}_{n-1}, t_{n-1}^+) e^{\frac{\imath H_S \tau}{\hbar}} M^\dagger(\xi_n)$$

*Can be also written as quantum state diffusion (QSD) for a pure state

$$M(\xi_n) \sim \prod_i \exp\left(\frac{\imath \gamma \tau \xi_{in} \hat{p}_i}{\hbar}\right) \exp\left(-\frac{(\xi_{in} - \hat{x}_i)^2}{2\Delta} \tau\right)$$

Momentum "feedback"

$$\gamma \sim \sqrt{\hbar/\Delta}$$

Caves and Milburn, Phys. Rev. A (1987)

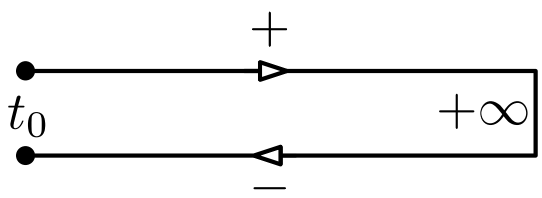
$$\Delta = \sigma\tau$$

Measurement strength Δ^{-1}

Limit of continuous weak measurement

$\sigma \to \infty, \tau \to 0$ with Δ finite

Schwinger-Keldysh path integral

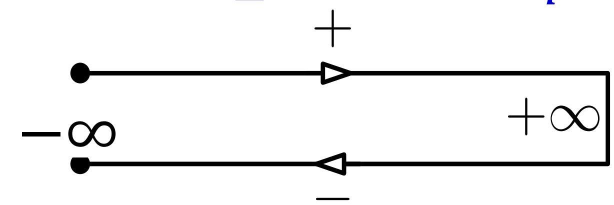


$$Tr[\rho(\{\xi(t)\})] = \int \mathcal{D}x \, e^{\frac{\imath S[\{\xi(t)\}, x(t)]}{\hbar}}$$

Action

$$S[\{\xi\},x] = \int_{-\infty}^{\infty} dt \sum_{s=\pm} s\left[\sum_i \left\{\frac{m}{2}\ \dot{x}_{is}^2 + m\gamma\ \dot{x}_{is}\xi_i + \frac{\imath s\hbar}{2\Delta}(x_{is}-\xi_i)^2\right\} - V(\{x_{is}\})\right]$$

Classical $(x_{ic} \equiv x_i)$ and quantum (x_{iq}) components $\quad x_{i\pm} = x_i \pm x_{iq}$



<u>Semiclassical limit, small $\hbar$</u>

Expand in x_{iq} or $\hbar$ keeping $\mathcal{O}\left(\frac{1}{\sqrt{\hbar}}\right), \mathcal{O}(1)$ while scaling $\Delta \sim \hbar^2$

$\Rightarrow$ Stochastic Langevin equation

$$\frac{d^2x_i}{dt^2} + \gamma\frac{dx_i}{dt} = \frac{1}{m}\left[-\frac{\partial V(\{x_i\})}{\partial x_i} + \eta_i(t)\right]$$

$$\langle \eta_i(t)\eta_j(t')\rangle = 2m\gamma T_{eff}\delta_{ij}\delta(t-t')$$

Noise strength $\sim \gamma T_{eff} \sim \frac{\hbar^2}{\Delta}$

$\propto$ measurement strength

Effective temperature $T_{eff} \sim \frac{\hbar^{\frac{3}{2}}}{\sqrt{\Delta}} \sim \sqrt{\hbar}$

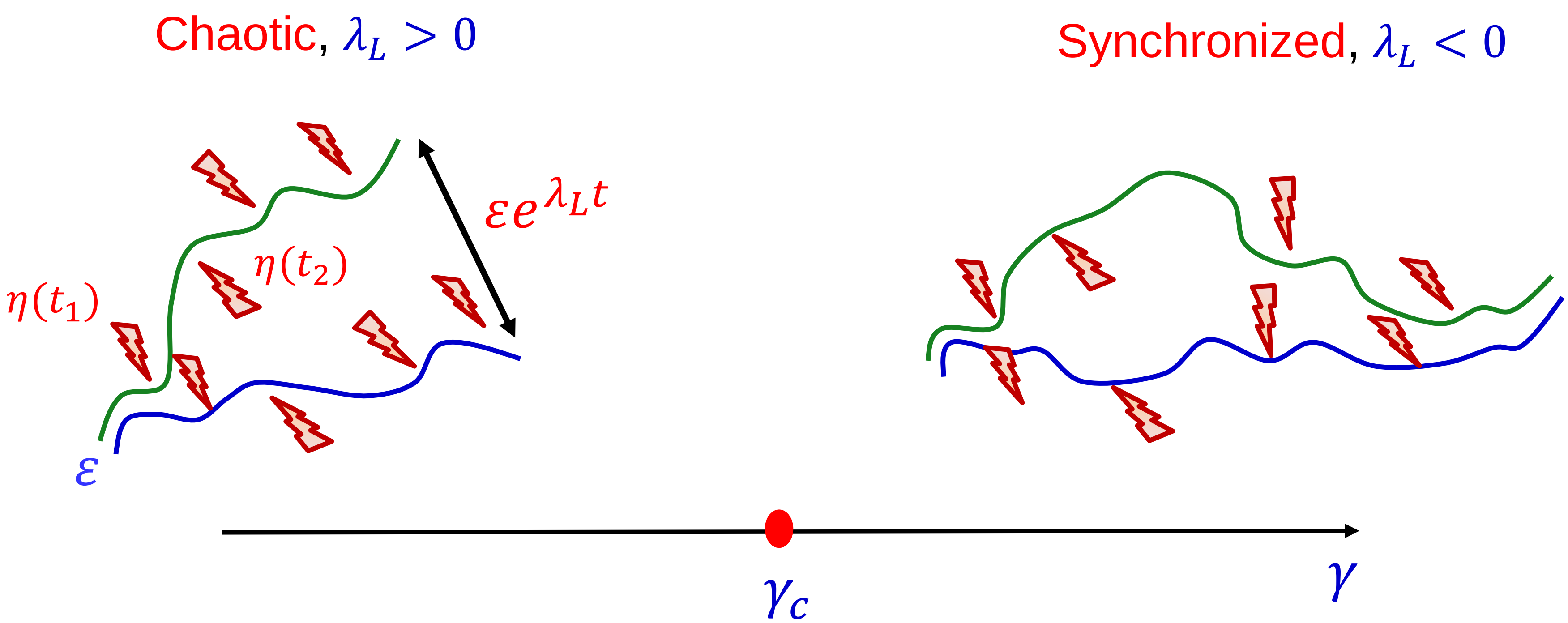
Long-time steady state (non-equilibrium pure steady state) $\Rightarrow$

Effective classical Boltzmann distribution $\sim \exp\left[-\frac{H_s(\{x_i,p_i\})}{T_{eff}}\right]$ for x and p

Can there be a dynamical phase transition with noise (measurement) strength in Langevin time evolution?

⇒ A "measurement induced phase transition (MIPT)" in the semiclassical limit

Yes, dynamical transition in many-body chaos



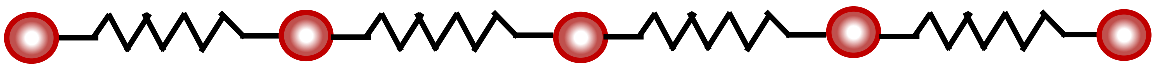
Noise/dissipation (measurement) strength

Integrable and non-integrable anharmonic chains of oscillators

1D chain, Langevin dynamics

$$\frac{d^2x_i}{dt^2} + \gamma \frac{dx_i}{dt} = \frac{1}{m}\left[-\frac{\partial V(\{x_i\})}{\partial x_i} + \eta_i(t)\right]$$

Noise strength, $\gamma \propto$ measurement strength



Two models:

1. Non-integrable model
   Anharmonic coupled oscillators

$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1} - x_i)^2 + \frac{u}{4}(x_{i+1} - x_i)^4\right]$$

2. Integrable model
   Toda chain

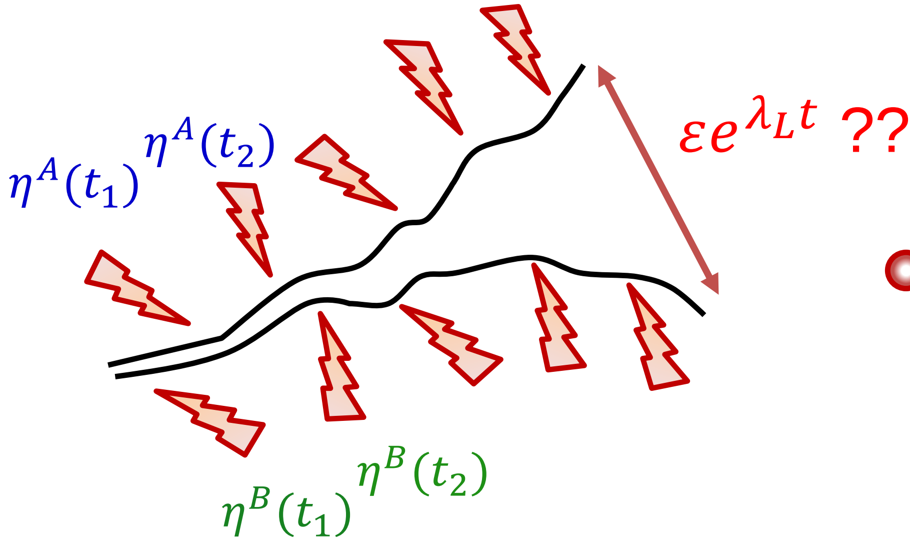
$$V(\{x_i\}) = \sum_i \left[\frac{a}{b} e^{-b(x_{i+1} - x_i)} + a(x_{i+1} - x_i) - \frac{a}{b}\right]$$

N constants of motion

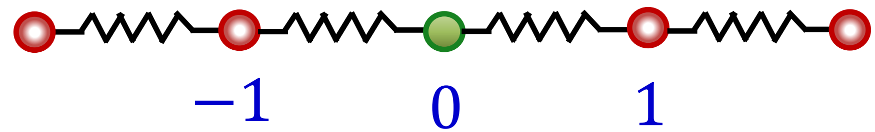
Harmonic limit $a \to \infty, b \to 0$; Hard sphere limit $b \to \infty$

Can one meaningfully define a classical OTOC in the presence of noise?

System is randomly kicked at each instant of time.



Noise strength, $\gamma \neq 0$



$$x_i^A(0) - x_i^B(0) = \varepsilon \delta_{i,0}$$

Take exactly the same noise realizations for the two copies

$$\{\eta_i^A(t)\} = \{\eta_i^B(t)\} \quad \forall t$$

Momentum OTOC

$$D(i,t) = \left\langle \left(p_i^A(t) - p_i^B(t) \right)^2 \right\rangle_{T,\{\eta\}}$$

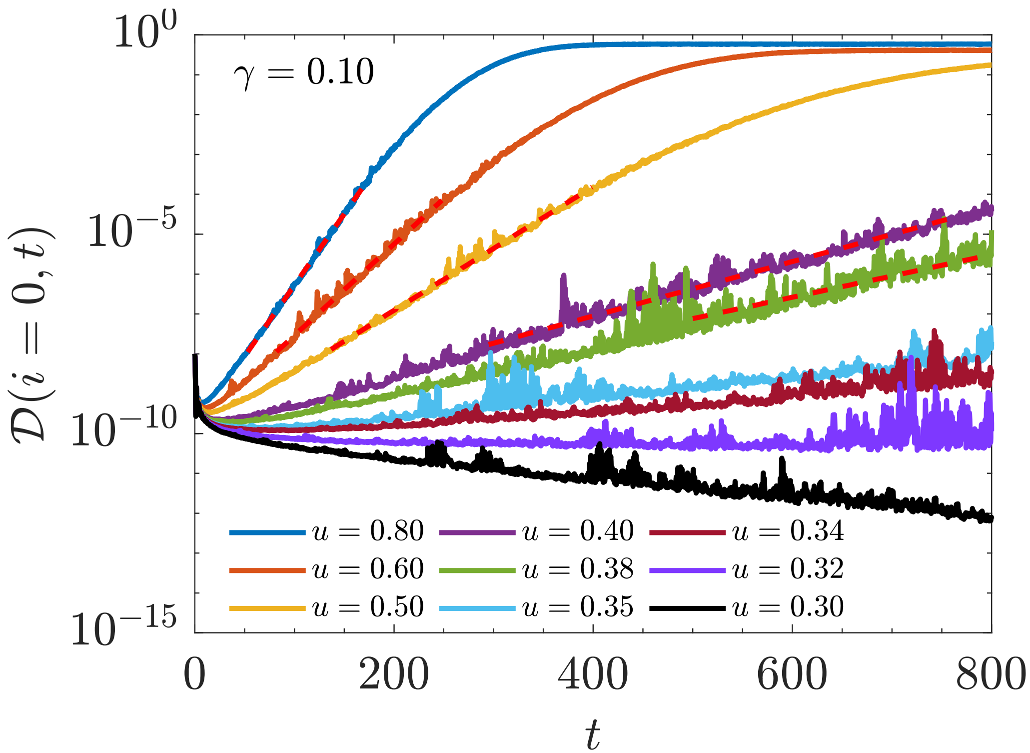
with perturbation at $i = 0, t = 0$

- Thermal initial condition at temperature T is generated using Langevin dynamics

Noise-induced chaotic to non-chaotic transition

Non-integrable model, Anharmonic coupled oscillators

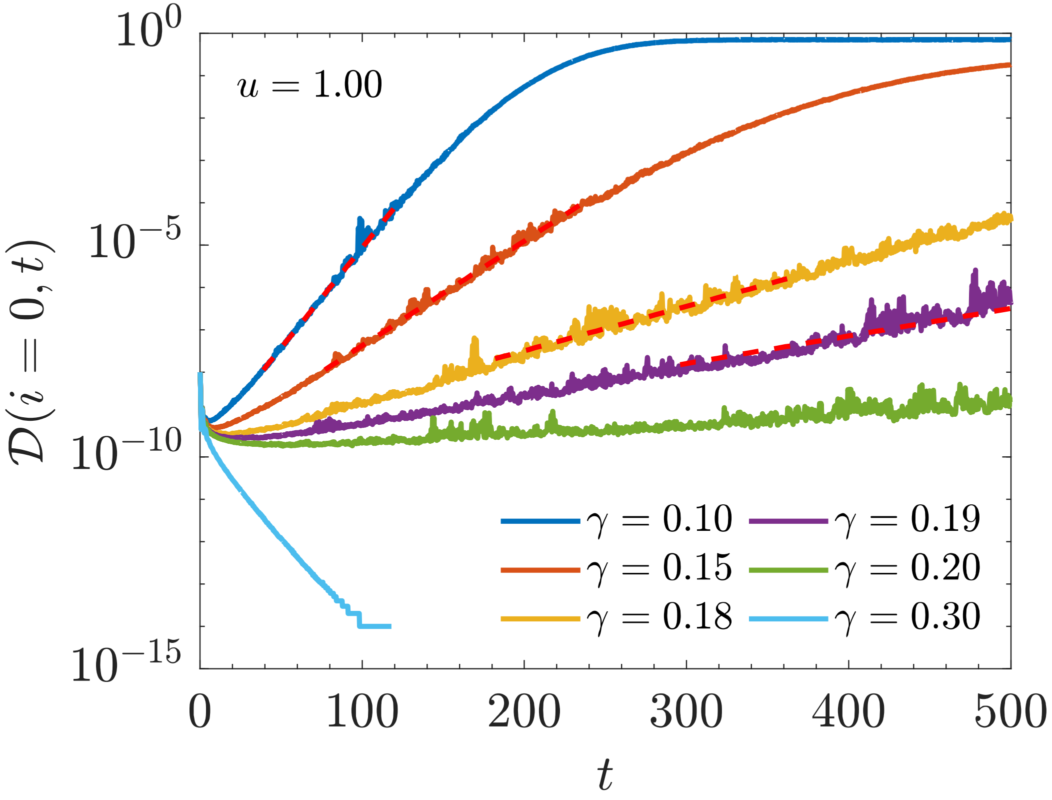
$$V(\{x_i\}) = \sum_i \left[\frac{k}{2}(x_{i+1} - x_i)^2 + \frac{u}{4}(x_{i+1} - x_i)^4\right]$$



- Harmonic limit $(u = 0)$ is non-chaotic.
- Transition from exponential growth to exponential decay as a function of decreasing u or u/γ

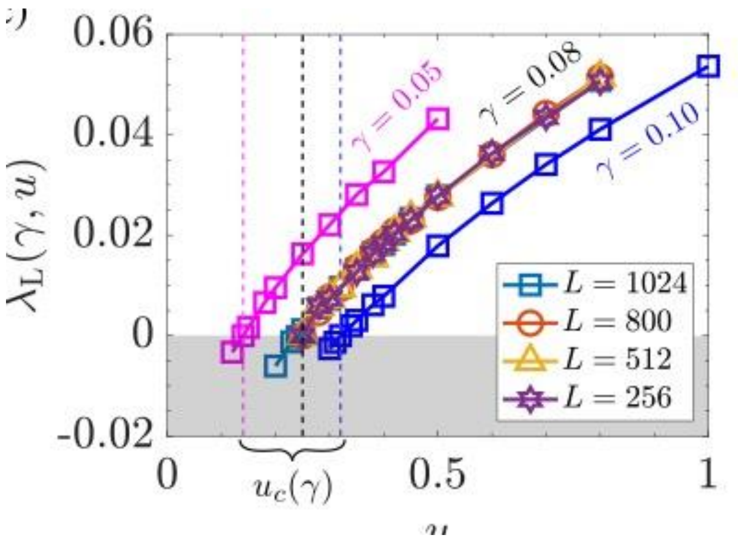
$$\lambda_L > 0 \quad \rightarrow \quad \lambda_L < 0$$

Transition as a function of γ

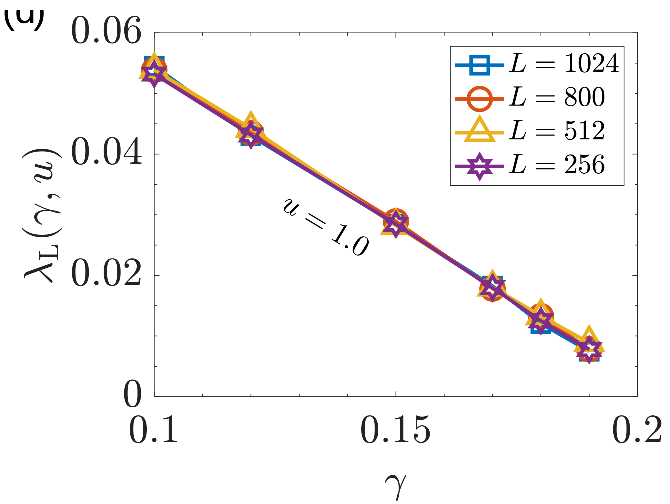


Lyapunov exponent

Transition as a function of u for fixed γ



Transition as a function of γ for fixed u

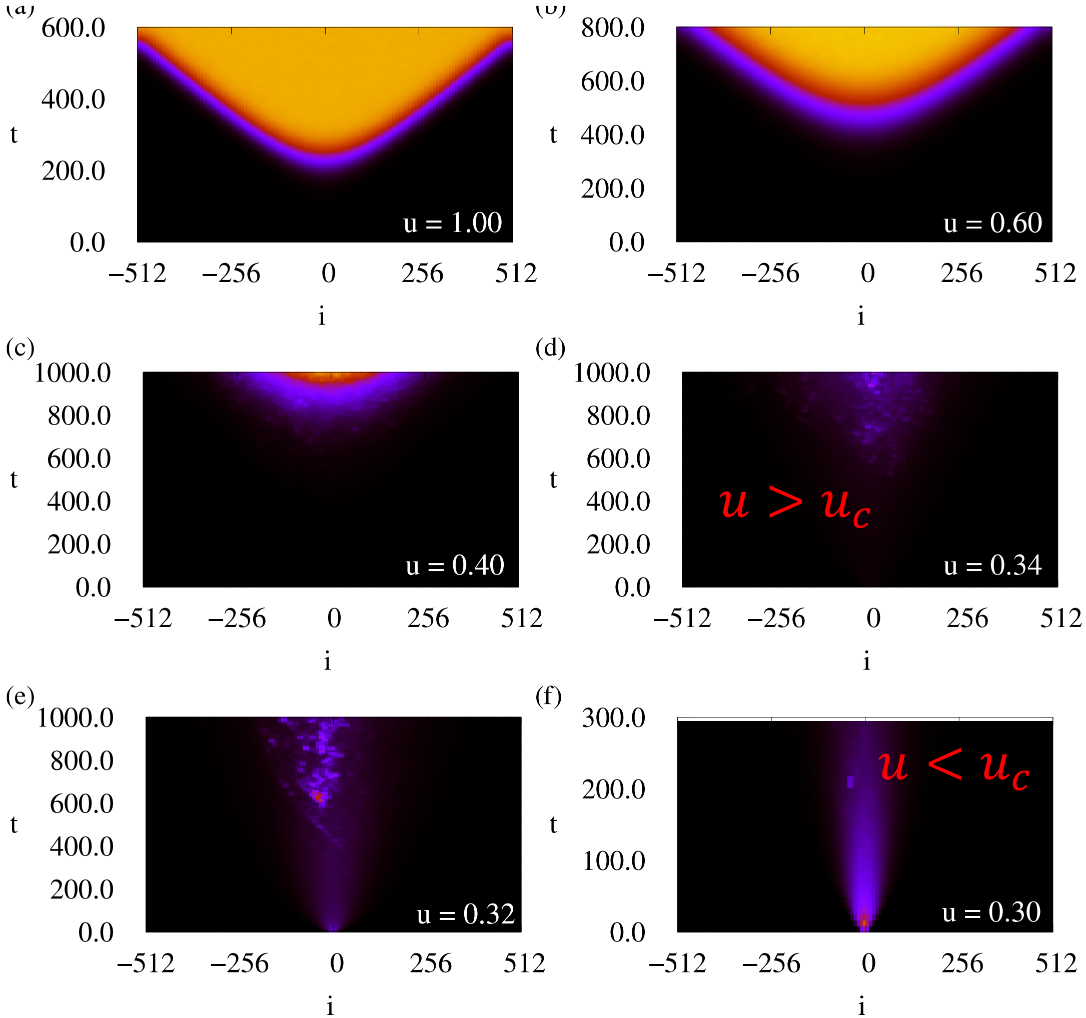


○ $\lambda_L > 0 \rightarrow \lambda_L < 0$ for $u < u_c(\gamma)$,

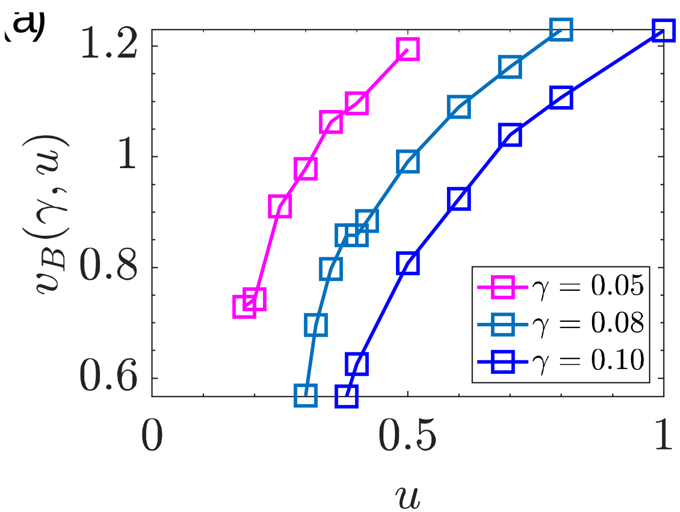
○ $\lambda_L > 0 \rightarrow \lambda_L < 0$ for $\gamma < \gamma_c(u)$,

* No system-size dependence in λ_L

Light cone and butterfly velocity



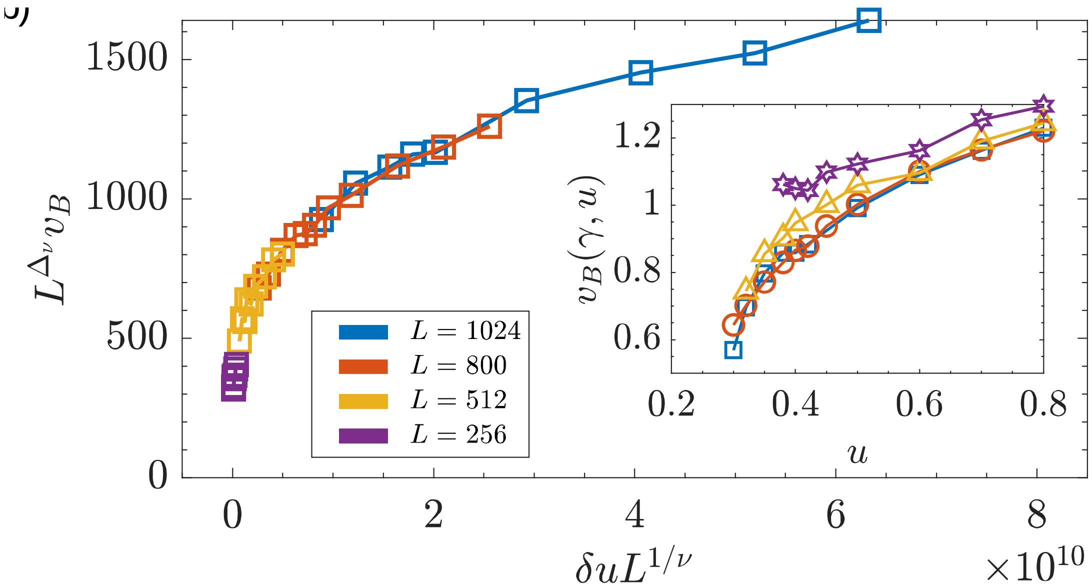
Butterfly velocity



- $v_B \to 0$ for $u \lesssim u_c(\gamma)$.

- Light cone is destroyed for $u < u_c(\gamma)$.

Dynamical transition and finite-size scaling



$\delta u = u - u_c > 0$

$v_B(u, L) = L^{-\frac{\beta}{\nu}} \mathcal{F}(\delta u \, L^{1/\nu})$

$v_B \sim (\delta u)^{\beta}$
$\xi \sim (\delta u)^{-\nu}$

$\beta \simeq 0.28, \qquad \nu \simeq 0.3$

- The transition shows critical scaling.

- The critical exponents do not match with known universality classes like directed percolation (DP) or multiplicative noise (MN)

Recent works on chaotic transition in classical systems
Willsher et al. PRB (2022); Deger et al. PRLs (2022)
- DP universality class

What is this chaotic-non chaotic transition?

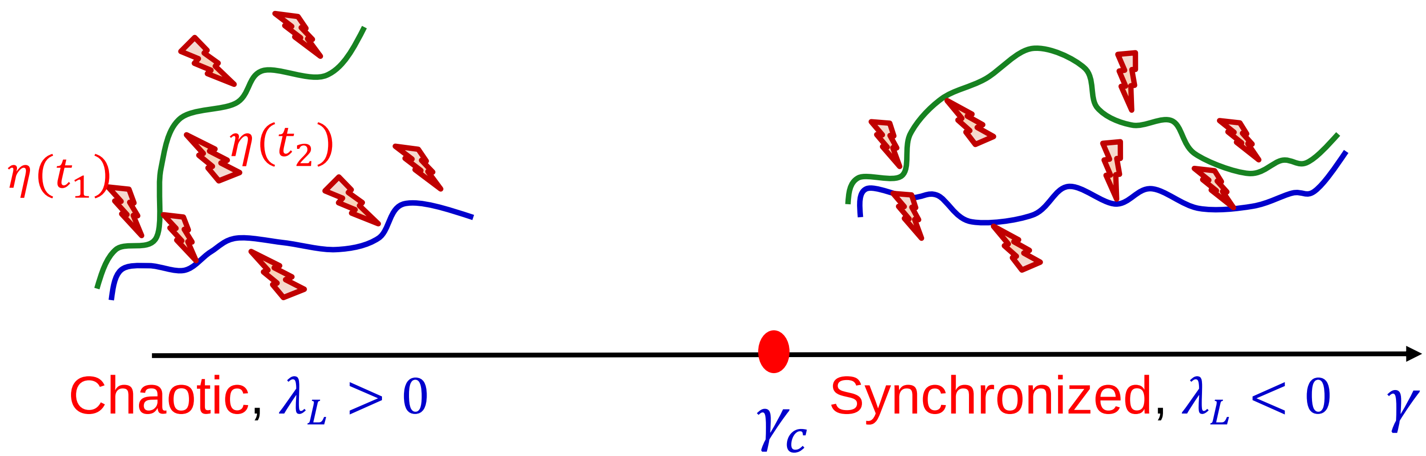
Stochastic synchronization transition (ST) in extended systems

Coupled map lattices (CML)

Bagnoli et al. PRE (1999); Baroni et al. PRE (2001); Cencini et al. PRE (2001); Ginelli et al. PRE (2003), ...

Multiplicative noise/KPZ and Directed percolation universality classes

Ahlers and Pikovsky, PRL (2002); Munoz et al. PRL (2003); ...



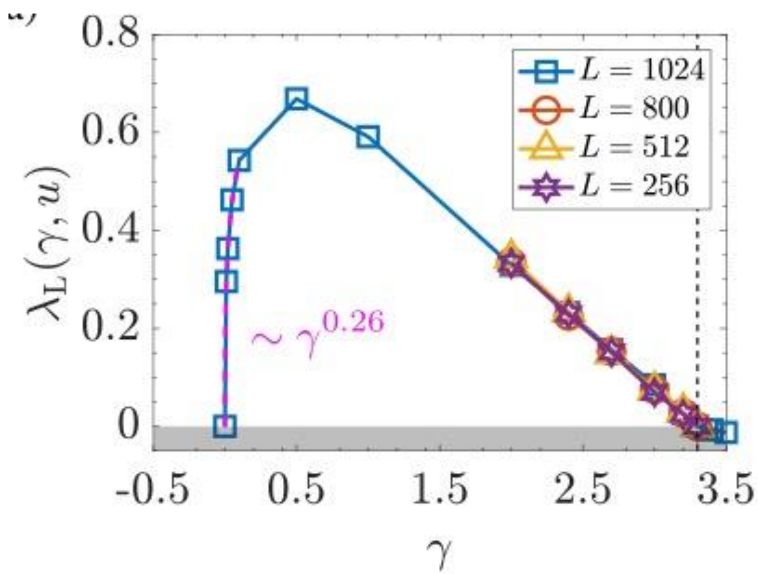
Noise/dissipation (measurement) strength

Stochastic ST as an MIPT in the semiclassical limit

Noise-induced chaotic to non-chaotic transitions in Toda chain

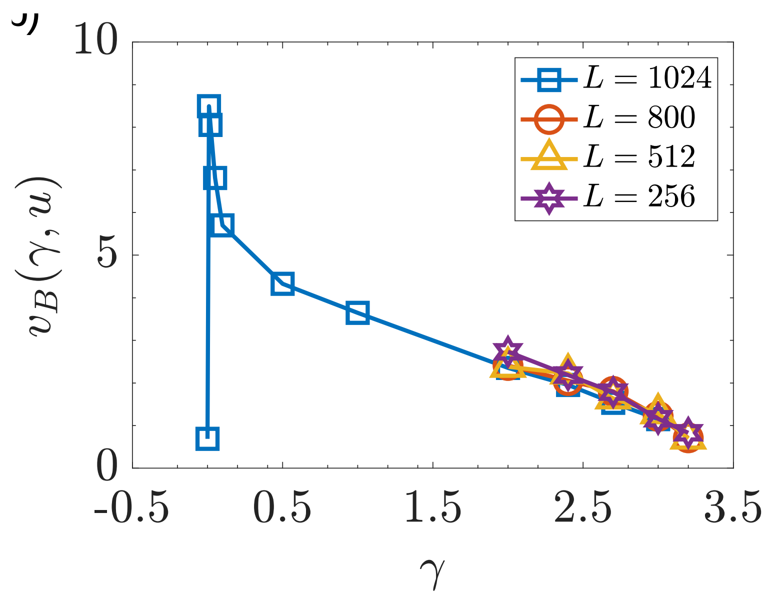
Integrable model $V(\{x_i\}) = \sum_i \left[\frac{a}{b} e^{-b(x_{j+1}-x_j)} + a(x_{j+1} - x_j) - \frac{a}{b}\right]$

Lyapunov exponent



- $\lambda_L \to 0$, $v_B \to$ large in the integrable limit $\gamma \to 0$.
- $\lambda_L, v_B \to 0$ for $\gamma > \gamma_c$.

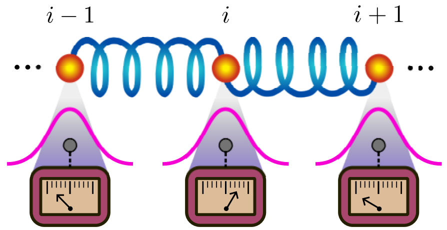
Butterfly velocity



- Weak noise induces weak chaos in integrable model

Lam and Kurchan, J. Stat. Phys. 156 (2014)

Summary and conclusion



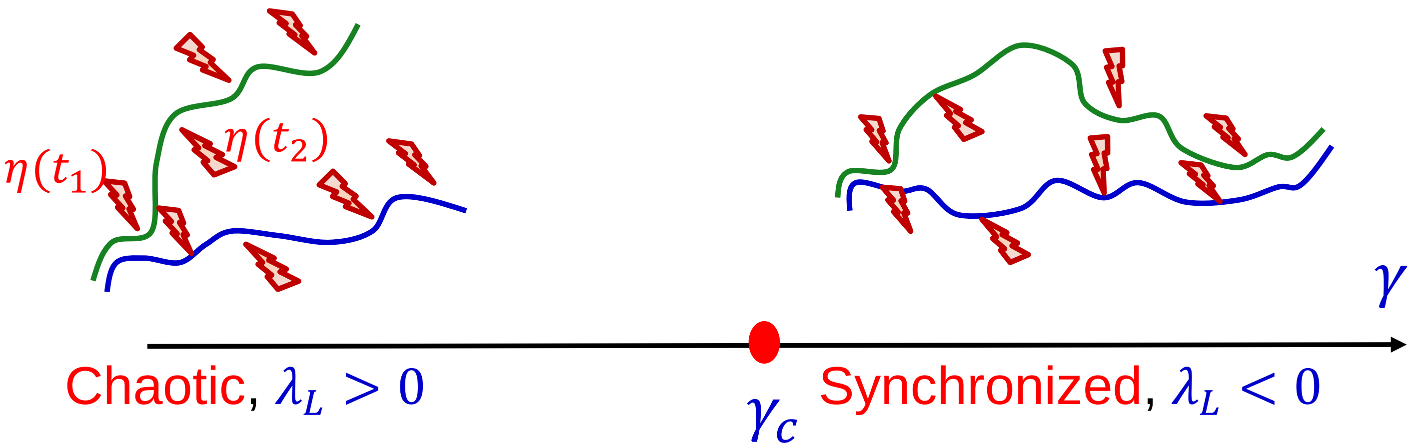
○ Semiclassical limit of a model of continuous weak measurements

$\Rightarrow$ Stochastic Langevin equation

noise/dissipation ∝ "measurement strength"

○ Noise/measurement induced chaotic to non-chaotic transition

Stochastic synchronization transition

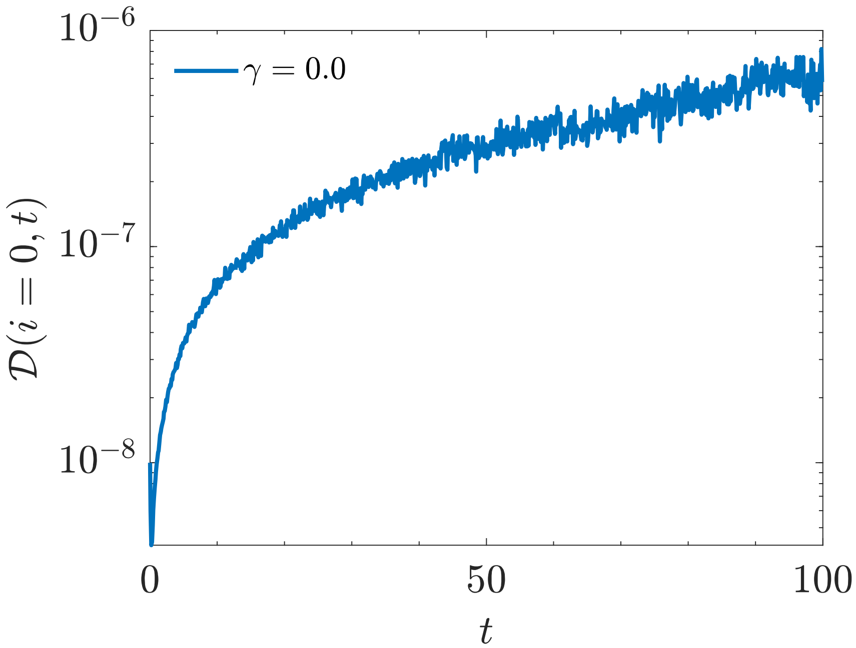


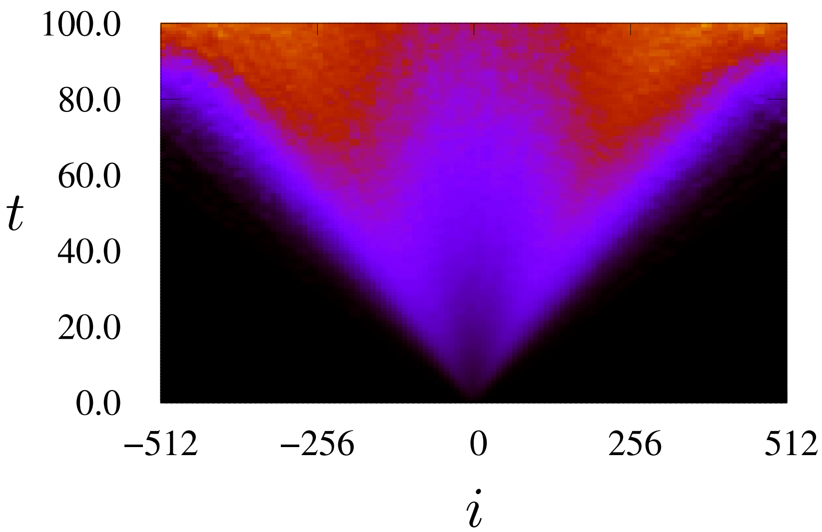
Noise/dissipation (measurement) strength

Thank You!

Many-body chaos in integrable Toda chain

$a = 0.07, b = 15$

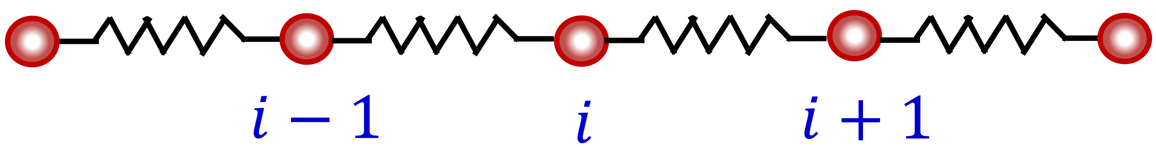


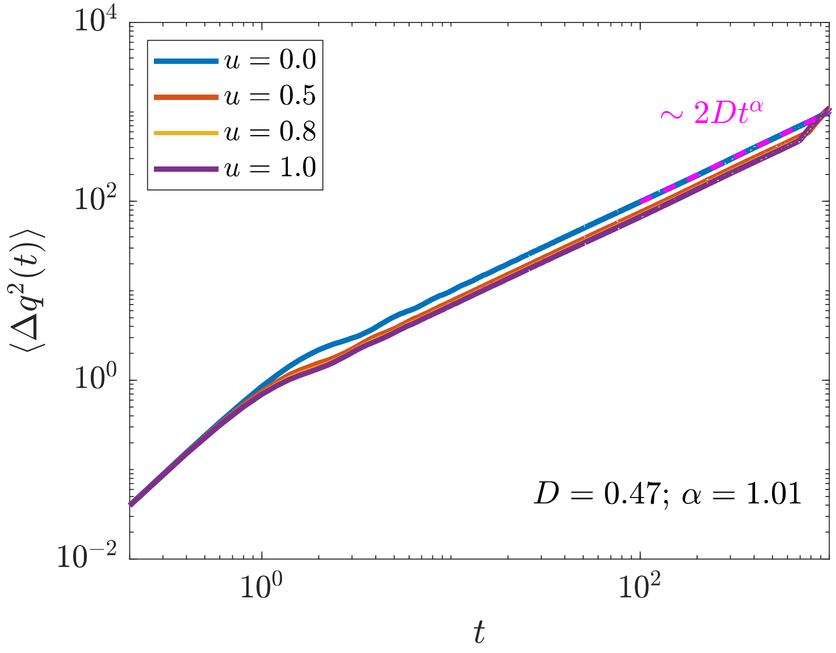


Light cone spread
with butterfly velocity
$v_B \neq 0$

- No exponential growth ($\lambda_L = 0$) in the integrable Toda chain.
- Non zero butterfly velocity

Is the transition visible in usual dynamical properties?





Diffusive for $\gamma = 0$

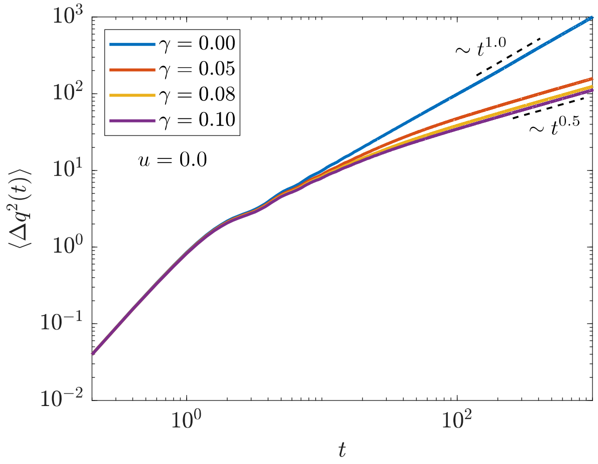
$u = 0$ (harmonic limit)

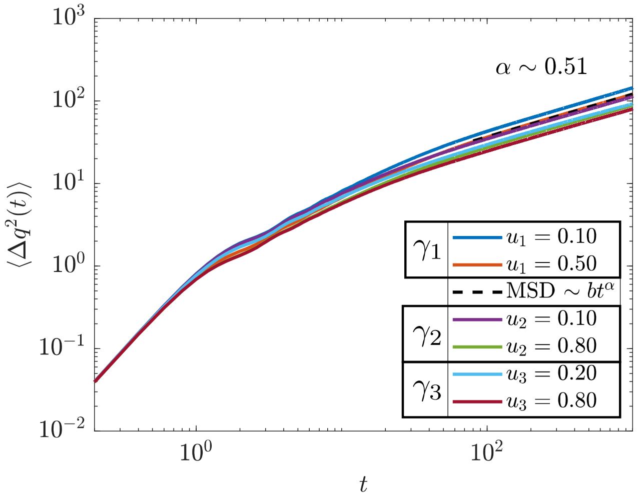
Diffusion constant

$$D = \frac{T}{2}\sqrt{\frac{1}{mk}}$$

Florencio and Lee, Phys. Rev. A 31 (1985)

Unlike chaos, there is no transition in usual dynamical properties, e.g. diffusion





$\gamma \neq 0$, subdiffusion

Monomer subdiffusin in ploymers
e.g. Weber et al., Phys. Rev. E 82 (2010)

Arguments for the existence of chaos bound

The proof for the bound, $\lambda_L \leq 2\pi k_B T/\hbar$, is not a rigorous proof!

- Maldacena-Shenker-Satnford $\Rightarrow$ Analytical properties of regularized OTOC + some physical assumptions

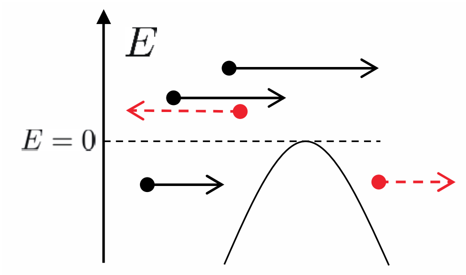
$$F(t) = \frac{1}{Z} Tr[e^{-\frac{\beta H}{4}} A(t) e^{-\frac{\beta H}{4}} B(0) e^{-\frac{\beta H}{4}} A(t) e^{-\frac{\beta H}{4}} B(0)]$$

- Energy-time uncertainty type argument (very crude)

$$\lambda_L^{-1} k_B T \geq \hbar$$

- Murthy and Srednicki, PRL (2019) $\Rightarrow$ Eigenstate thermalization hypothesis + assumptions.

- Morita, SciPost (2021) $\Rightarrow$ Effective model for classical system with Lyapunov exponent $\rightarrow$ inverse Harmonic potential



$$P(E) := \frac{1}{\exp(\beta_L |E|) + 1}$$

Analogous Hawking radiation temperature

$$T_L := \frac{1}{\beta_L} = \frac{\hbar}{2\pi} \lambda_L$$

Thermal equilibrium $\Rightarrow T \geq T_L \Rightarrow \lambda_L \leq 2\pi k_B T/\hbar$

Other quantities related to OTOC, quantum chaos, operator and/or entanglement growth, thermalization

Loschmidt echo

Kurchan (2017)

Fidelity for 'kicked' perturbation

$$F = \mathrm{Tr}[A_{tran}A] = \mathrm{Tr}\left\{\left[e^{i\frac{t}{\hbar}H}e^{i\frac{\delta}{\hbar}B}e^{-i\frac{t}{\hbar}H}\right]A\left[e^{i\frac{t}{\hbar}H}e^{i\frac{\delta}{\hbar}B}e^{-i\frac{t}{\hbar}H}\right]^{\dagger}A\right\}$$

Choose $B(t) = e^{i\frac{t}{\hbar}H}Be^{-i\frac{t}{\hbar}H}$

$$Tr[A^2] - Tr[A_{tran}A] = -\frac{\delta^2}{2\hbar^2}Tr([B(t),A(0)]^2) + O(\delta^3)$$

1. Loschmidt echo $A = |\psi\rangle\langle\psi| \quad \Rightarrow \quad F = \left|\left\langle\psi\left|e^{\frac{it}{\hbar}H}e^{\frac{i\delta}{\hbar}B}\,e^{-\frac{it}{\hbar}H}\right|\psi\right\rangle\right|^2$

2. OTOC $A \propto e^{-\frac{\beta H}{4}}Ae^{-\frac{\beta H}{4}} \quad \Rightarrow \quad Tr([B(t),A(0)]^2) = F_{MS}(t)$