Generalized multifractality at Anderson-Mott transitions

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Introduction:

- Anderson model and transitions
- symmetry classes
- generalized multifractality

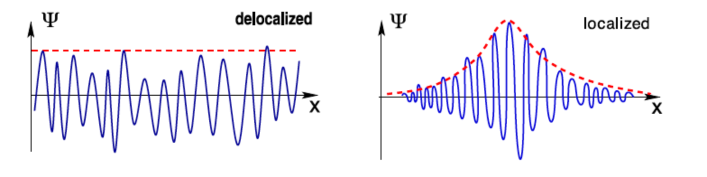
[Anderson (1958); Abrahams et al. (1979)]

- Anderson model on lattice

$$H = -\sum_{<i,j>} (a_i^\dagger a_j + a_j^\dagger a_i) + \sum_i \epsilon_i a_i^\dagger a_i, \quad \epsilon_i \in [-W/2, W/2]$$

- all states are localized in $d = 1$
- metalic to localization crossover in $d = 2$
- Anderson transition between delocalized and localized states in $d = 3$

$W < W_c$ $\qquad\qquad$ $W > W_c$



courtesy of P. Nosov

[Anderson (1958); Abrahams et al. (1979)]

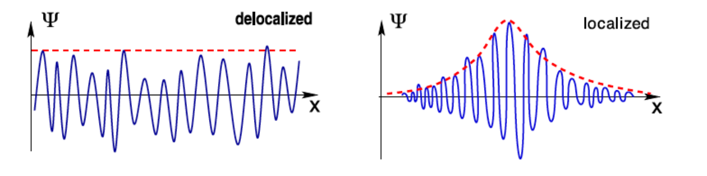
- a single electron in a random potential

$$H = -\frac{\nabla^2}{2m} + V(\boldsymbol{r}), \qquad H\psi = E\psi$$

- Anderson transition is a continuous quantum phase transition with divergent correlation length at the mobility edge,

$$\xi \sim |E - E_c|^{-\nu}$$

$E < E_c$ $E > E_c$



courtesy of P. Nosov

[Zirnbauer (1996); Altland, Zirnbauer (1997)]

- random single-particle Hamiltonians with/without spin rotational, time reversal and chiral symmetries can be classified in accordance with classification of symmetric spaces

Ham. class	RMT	T	S	compact symmetric space	non-compact symmetric space	σ-model B|F	σ-model compact sector $\mathcal{M}_F$
Wigner-Dyson classes							
A	GUE	$-$	$\pm$	$U(N)\times U(N)/U(N)\equiv U(N)$	$GL(N,\mathbb{C})/U(N)$	AIII|AIII	$U(2n)/U(n)\times U(n)$
AI	GOE	$+$	$+$	$U(N)/O(N)$	$GL(N,\mathbb{R})/O(N)$	BDI|CII	$Sp(4n)/Sp(2n)\times Sp(2n)$
AII	GSE	$+$	$-$	$U(2N)/Sp(2N)$	$U^*(2N)/Sp(2N)$	CII|BDI	$O(2n)/O(n)\times O(n)$
chiral classes							
AIII	chGUE	$-$	$\pm$	$U(p+q)/U(p)\times U(q)$	$U(p,q)/U(p)\times U(q)$	A|A	$U(n)$
BDI	chGOE	$+$	$+$	$SO(p+q)/SO(p)\times SO(q)$	$SO(p,q)/SO(p)\times SO(q)$	AI|AII	$U(2n)/Sp(2n)$
CII	chGSE	$+$	$-$	$Sp(2p+2q)/Sp(2p)\times Sp(2q)$	$Sp(2p,2q)/Sp(2p)\times Sp(2q)$	AII|AI	$U(n)/O(n)$
Bogoliubov - de Gennes classes							
C		$-$	$+$	$Sp(2N)\times Sp(2N)/Sp(2N)\equiv Sp(2N)$	$Sp(2N,\mathbb{C})/Sp(2N)$	DIII|CI	$Sp(2n)/U(n)$
CI		$+$	$+$	$Sp(2N)/U(N)$	$Sp(2N,\mathbb{R})/U(N)$	D|C	$Sp(2n)$
BD		$-$	$-$	$SO(N)\times SO(N)/SO(N)\equiv SO(N)$	$SO(N,\mathbb{C})/SO(N)$	CI|DIII	$O(2n)/U(n)$
DIII		$+$	$-$	$SO(2N)/U(N)$	$SO^*(2N)/U(N)$	C|D	$O(n)$

[adopted from Evers, Mirlin (2008)]

Multifractality: moments of the local density of states

[Wegner (1980,1987); Kravtsov, Lerner (1985); Pruisken(1985); Castellani, Peliti (1986)]

- local density of states (LDoS) in the cube of size L

$$\rho(E,\boldsymbol{r}) = \sum_{\alpha} \left|\psi_\alpha(\boldsymbol{r})\right|^2 \delta(E-\epsilon_\alpha)$$

where $\psi_\alpha(\boldsymbol{r})$ and ϵ_α w. f. and energy for a given disorder realization

- scaling of the moments of LDoS

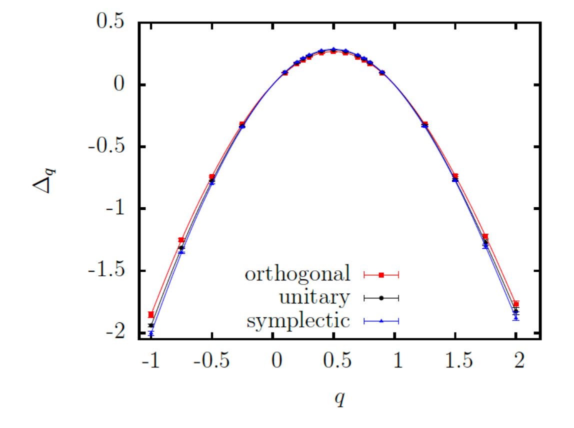
$$\left\langle [\rho(E,\boldsymbol{r})]^q \right\rangle_{\text{dis}} \sim L^{-x_{(q)}}, \qquad q = 0,1,2,\ldots, \quad x_{(q)} = q x_1 + \Delta_q$$

- normaization conditions: $\Delta_0 = \Delta_1 = 0$
- naive expectation for a metal: $x_1 = \Delta_q = 0$
- the exponent $\Delta_q \leqslant 0$ is nonlinear function of q

[for a review, see Evers&Mirlin (2008)]

NB the statistics of energy levels ϵ_α (global density of states) across the Anderson transition is not discussed in this talk

- multifractal exponents at $d = 3$ Anderson transition from numerical calculations

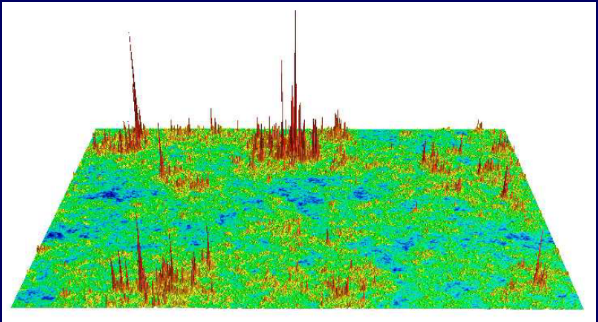


[Ujfalusi, Varga (2015)]

NB orthogonal – AI, unitary – A, symplectic – AII

Multifractality: physical meaning

- let α be a random number such that $|\psi_\alpha(\boldsymbol{r})|^2 \sim L^{-a}$ then its distribution (i.e. distribution of w.f.) is given as $\mathcal{P}(a) \sim L^{f(a)}$



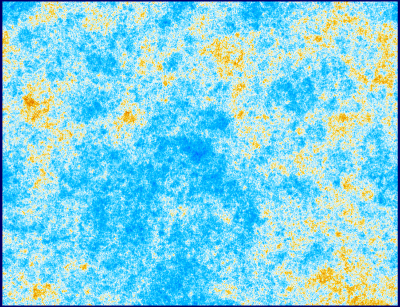
[courtesy by Evers, Mildenberger, Mirlin]

- the singularity spectrum function is related by Legendre transform with $\tau_q = d(q-1) + \Delta_q$

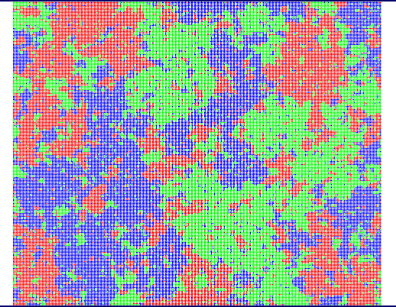
$$f(\alpha) = \alpha q - \tau_q, \qquad \alpha = \frac{d\tau_q}{dq}, \qquad q = \frac{df(\alpha)}{d\alpha}$$

◦ comparison of wave function at Anderson transition and spin in the 3-state critical Potts model

multifractal



fractal



[courtesy by M. Puschmann and M. Lehmkühler]

Generalized multifractality: scaling of operators

- an infinite set of local pure scaling operators

$$O_\lambda \sim L^{-x_\lambda}$$

[Höf&Wegner (1986)]

where $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_k$ is the set of partitions of an integer $|\lambda| = \lambda_1 + \cdots + \lambda_k$.

- different xponents x_λ are related by Weyl symmetry, $x_\lambda = x_{w\lambda}$

[Gruzberg, Ludwig, Mirlin, Zirnbauer (2011), Gruzberg, Mirlin, Zirnbauer (2013), Karcher, Charles, Gruzberg, Mirlin (2021), Karcher, Gruzberg, Mirlin (2022)]

- simplest example of operators (class A)

$$L^{2d}\langle\psi_\alpha(\boldsymbol{r})|^4\rangle_{\text{dis}} \sim L^{-x_{(2)}},\ L^{2d}\left\langle\left|\psi_\alpha(\boldsymbol{r_1})\psi_\beta(\boldsymbol{r_2}) - \psi_\beta(\boldsymbol{r_1})\psi_\alpha(\boldsymbol{r_2})\right|^2\right\rangle_{\text{dis}} \sim L^{-x_{(1,1)}}$$

- examples of Weyl symmetry relations,

$x_{(q)} = x_{(1-q)}$ and $x_{(q_1,q_2)} = x_{(1-q_1,q_2)} = x_{(q_1,3-q_2)}$ for class A
$x_{(q)} = x_{(3-q)}$ and $x_{(q_1,q_2)} = x_{(3-q_1,q_2)} = x_{(q_1,7-q_2)}$ for class C

- Anderson transitions in $d = 2$ occur at strong coupling
- conformal field theory is natural suggestion for a critical theory at Anderson transition
- assumption of local conformal invariance and abelian fusion leads to "parabolic" generalized multicritical exponents:

$$x_{\lambda}^{(CFT)} = -b\lambda(\lambda + \rho_b)$$

where a vector ρ_b is specific for a particular symmetry class whereas b is specific for a particular CFT.

[Bondesan, Wieczorek, Zirnbauer (2017); Karcher, Charles, Gruzberg, Mirlin (2021)]

- example: 2D Anderson transition in class A (integer quantum Hall effect)

$$x_{(q)}^{(CFT)} = -b_A q(q-1), \qquad x_{(q_1,q_2)}^{(CFT)} = -b_A[q_1(q_1-1) + q_3(q_3-3)]$$

- example: 2D Anderson transition in class A (integer quantum Hall effect)

$$x_{(q)}^{(CFT)} = b_C q(3-q), \quad x_{(q_1,q_2)}^{(CFT)} = b_C[q_1(3-q_1) + q_2(7-q_2)]$$

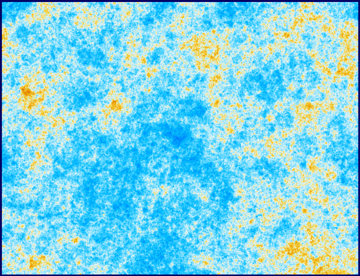
- generalized parabolicity for 2D Anderson transitions in classes A (integer qHe) and C (spin qHe) is not consistent neither with numerical results nor with analytics (for class C)

  [Karcher, Charles, Gruzberg, Mirlin (2021); Puschmann, Hernangómez-Pérez, Lang, Bera, Evers (2021)]

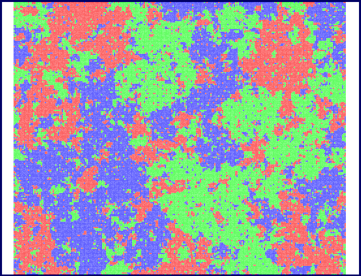
- in particular, it holds for Wess-Zumino- Novikov-Witten models suggested for the iqHe criticality

  [Zirnbauer (1999); Kettemann, Tsvelik (1999); Zirnbauer (2019)]

scale invariance

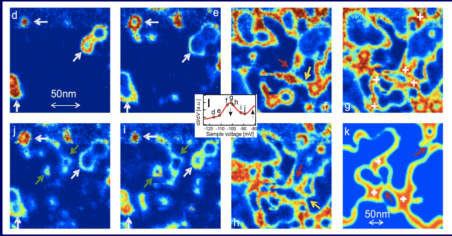


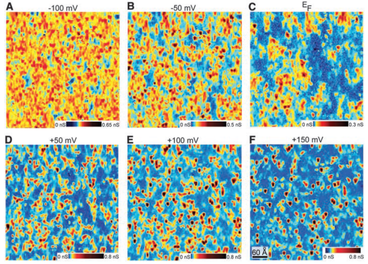
conformal invariance



[courtesy by M. Puschmann and M. Lehmkühler]

- differential conductance map in InSb at $B = 12$ T (left) and at surface of $Ga_{1-x}Mn_xAs$ with $x = 1.5$ (right) %





[Morgenstern et al. (2012); Richardella et al. (2010)]

Question to answer:

What's about electron-electron interaction?

How does electron-electron interaction affect generalized multifractality in class C?

Class C: spin qHe

- symmetries
  - time reversal symmetry - No
  - spin rotation symmetry - Yes
  - particle-hole (BdG) symmetry - Yes

- current of z-component of the spin along x direction in the presence Zeeman field along z direction

$$j_x^{(\mathrm{z})} = -\sigma_{xy}^{(\mathrm{s})} \frac{dB_z(y)}{dy}, \qquad \sigma_{xy}^{(\mathrm{s})} = \frac{n}{4\pi}$$

- possible physical relalization:
  $d_{x^2-y^2} + id_{xy}$ disordered superconductors

[Volovik JETP Lett. (1997), Kagalovsky et al., PRL (1999), Senthil, Marston, Fisher, PRB (1999)]

- pure scaling operators are straightforward generalization of pure scaling operators of noninteracting case
- pure scaling operators can be formulated in terms of Green's function correlations rather than wave functions
- symmetry relations for Δ_λ holds within second order in $\epsilon = d - 2$ expansion (two-loop approximation)

[Burmistrov, Gornyi, Mirlin (2013&2015), Repin, Burmistrov (2016)]

Formalism: field theory for class C

- Finkel'stein NLσM for class C

$$S = \frac{g}{16}\int_{\boldsymbol{x}} \mathrm{Tr}(\nabla Q)^2 - Z_\omega \int_{\boldsymbol{x}} \mathrm{Tr}\,\hat{\varepsilon} Q + \frac{\pi T \Gamma_t}{4} \sum_{\alpha,n} \int_{\boldsymbol{x}} \mathrm{Tr}(I_n^\alpha \vec{\boldsymbol{s}} Q)\,\mathrm{Tr}(I_{-n}^\alpha \vec{\boldsymbol{s}} Q)$$

[see Bruno, Toschi, Dell'Anna, Castellani (2005); Dell'Anna (2006)]

- matrix notations (n, m - Matsubara, α, β - replica)

$$(I_k^\gamma)_{nm}^{\alpha\beta} = \delta_{n-m,k}\delta^{\alpha\beta}\delta^{\alpha\gamma} s_0, \; \hat{\varepsilon}_{nm}^{\alpha\beta} = \varepsilon_n\,\delta_{nm}\delta^{\alpha\beta} s_0, \; \Lambda_{nm}^{\alpha\beta} = \mathrm{sgn}\,\varepsilon_n\,\delta_{nm}\delta^{\alpha\beta} s_0$$

- symmetries and constraints

$$Q^\dagger = Q, \quad \boxed{Q = -\bar{Q}}, \quad Q^2(\boldsymbol{x}) = 1,$$
$$\bar{Q} = s_y L_0 Q^T L_0 s_y, \quad (L_0)_{nm}^{\alpha\beta} = \delta_{\varepsilon_n, -\varepsilon_m}\delta^{\alpha\beta}$$

- target manifold

$$Q \in G/K = \mathrm{Sp}(2N)/\mathrm{U}(N), \qquad N = 2N_r N_m$$
$$\alpha = 1, \dots N_r, \quad \varepsilon_n = \pi T(2n+1), \quad n = -N_m, \dots, N_m - 1$$

Anderson-Mott transition: one-loop RG equations

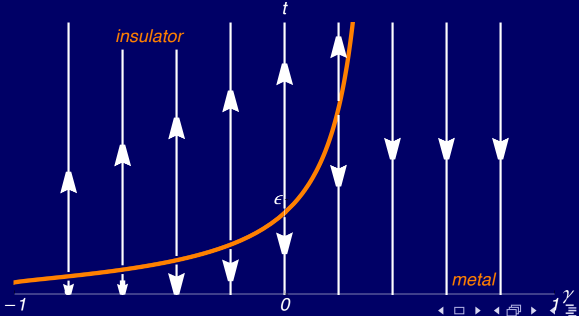
- RG equations in $d = 2 + \epsilon$ to the lowest order in $t = 1/(\pi g)$

$$\frac{dt}{d\ln L} = -\epsilon t + \left[1 + 6f(\gamma)\right]t^2 + O(t^3), \qquad \frac{d\gamma}{d\ln L} = 0 + O(t^2)$$

$$\frac{d\ln Z_\omega}{d\ln L} = -(1-3\gamma)t + O(t^2)$$

where $\gamma = \Gamma_t/Z_\omega$ and $f(\gamma) = 1-(1+1/\gamma)\ln(1+\gamma)$.

[Jeng, Ludwig, Senthil, Chamon (2001); Dell'Anna (2006); Liao, Levchenko, Foster (2017)]



- dimensionless interaction parameter $\gamma = \Gamma_t / Z_\omega$. It is not renormalized to the first order in γ at noninteracting fixed point

$$\frac{d\gamma}{d\ln L} = \underbrace{\left(x_{(1)} - x_{(2)}\right)}_{=0} \gamma + O(\gamma^2)$$

- local derivativeless operator with q matrices Q involved

$$\mathcal{K}_q(E_1,\ldots,E_q) = \frac{1}{4^q} \sum_{p_1,\ldots p_q=\pm} \left(\prod_{j=1}^{q} p_j\right) \mathcal{P}_q^{\alpha_1,\ldots,\alpha_q;p_1,\ldots,p_q}(E_1,\ldots,E_q),$$

where $\alpha_j \neq \alpha_k$ if $j \neq k$ for $j,k=1,\ldots,q$ and $\mathcal{P}_q^{\alpha_1,\ldots,\alpha_q;p_1,\ldots,p_q}$ is obtained by analytic continuation $\varepsilon_{n_j} \to E_j + ip_j 0^+$ from

$$P_q^{\alpha_1,\ldots,\alpha_q}(i\varepsilon_{n_1},\ldots,i\varepsilon_{n_q}) = \sum_{\{k_1,\ldots,k_s\}} \mu_{k_1,\ldots,k_s} R_{k_1,\ldots,k_s},$$

$$R_{k_1,\ldots,k_s} = \prod_{r=k_1}^{k_s} \operatorname{tr} Q^{\alpha_{j_1}\alpha_{j_2}}_{n_{j_1}n_{j_2}} Q^{\alpha_{j_2}\alpha_{j_3}}_{n_{j_2}n_{j_3}} \ldots Q^{\alpha_{j_r}\alpha_{j_1}}_{n_{j_r}n_{j_1}},$$

where $\{k_1,\ldots,k_s\}$ are partitions of q: $k_1+k_2+\ldots+k_s=q$ and $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_s > 0$

- Example:

$$P_2^{\alpha_1\alpha_2}(i\varepsilon_n, i\varepsilon_m) = \operatorname{tr} Q^{\alpha_1\alpha_1}_{nn}(\boldsymbol{r}) \operatorname{tr} Q^{\alpha_2\alpha_2}_{mm}(\boldsymbol{r}) + \mu_2 \operatorname{tr}\left[Q^{\alpha_1\alpha_2}_{nm}(\boldsymbol{r}) Q^{\alpha_2\alpha_1}_{mn}(\boldsymbol{r})\right]$$

more examples:

$$P_3 = \operatorname{tr} Q^{\alpha\alpha}_{kk} \operatorname{tr} Q^{\beta\beta}_{nn} \operatorname{tr} Q^{\mu\mu}_{mm} + \mu_{2,1} \operatorname{tr} Q^{\alpha\alpha}_{kk} \operatorname{tr} Q^{\beta\mu}_{nm} Q^{\mu\beta}_{mn} + \mu_3 \operatorname{tr} Q^{\alpha\beta}_{kn} Q^{\beta\mu}_{nm} Q^{\mu\alpha}_{mk}$$

$$P_4 = \operatorname{tr} Q^{\alpha\alpha}_{kk} \operatorname{tr} Q^{\beta\beta}_{nn} \operatorname{tr} Q^{\mu\mu}_{mm} \operatorname{tr} Q^{\nu\nu}_{ll} + \mu_{2,1,1} \operatorname{tr} Q^{\alpha\beta}_{kn} Q^{\beta\alpha}_{nk} \operatorname{tr} Q^{\mu\mu}_{mm} \operatorname{tr} Q^{\nu\nu}_{ll} + \mu_{3,1} \operatorname{tr} Q^{\alpha\beta}_{kn} Q^{\beta\mu}_{nm} Q^{\mu\alpha}_{mk} \operatorname{tr} Q^{\nu\nu}_{ll} + \mu_{2,2} \operatorname{tr} Q^{\alpha\beta}_{kn} Q^{\beta\alpha}_{nk} \operatorname{tr} Q^{\mu\nu}_{ml} Q^{\nu\mu}_{lm} + \mu_4 \operatorname{tr} Q^{\alpha\beta}_{kn} Q^{\beta\mu}_{nm} Q^{\mu\nu}_{ml} Q^{\nu\alpha}_{lk}$$

NB in the noninteracting case one can average over global $U(N)$ rotations, $Q \to R^{-1} Q R$ with $[R, \Lambda] = 0$

Results: pure scaling operators / anomalous dimensions

- renormalization of the pure scaling operator with q matrices Q:

$$\langle \mathcal{K}_q \rangle = Z^q M_q \mathcal{K}_q[\Lambda]$$

- LDoS renormalization factor within one-loop approximation ($\langle Q \rangle = Z\Lambda$):

$$\eta_{(1)} = -\frac{d \ln Z}{d \ln L} = [1 - 3\ln(1+\gamma)]t + O(t^2)$$

- anomalous dimensions within two-loop approximation

$$\eta^{(\mu_{2,1,\dots,1})} = -\frac{d \ln M_q}{d \ln L} = \mu_{2,1,\dots,1} t[1 + 3c(\gamma)t] + O(t^3),$$
$$c(\gamma) = 2 + \frac{1+\gamma}{2\gamma} \ln^2(1+\gamma) + \frac{2+\gamma}{\gamma} \operatorname{li}_2(-\gamma)$$

- coefficient $\mu_{2,1,\dots,1}$ is the same as in noninteracting case

$$\mu_{2,1,\dots,1} = \frac{1}{2} \sum_{j=1}^{s} k_j(-c_j - 1 - k_j), \quad c_j = 1 - 4j.$$

[Karcher, Charles, Gruzberg, Mirlin (2021)]

Results: examples of pure scaling operators for class C

$q = 2$

λ	μ_2
(2)	-1
(1,1)	2

$q = 3$

λ	$\mu_{2,1}$	μ_3
(3)	-3	2
(2,1)	1	-2
(1,1,1)	6	8

$q = 4$

λ	$\mu_{2,1,1}$	$\mu_{3,1}$	$\mu_{2,2}$	μ_4
(4)	-6	8	3	-6
(3,1)	-1	-2	-2	4
(2,2)	2	-8	7	-2
(2,1,1)	5	4	-2	-8
(1,1,1,1)	12	32	12	48

[cf. Karcher, Charles, Gruzberg, Mirlin (2021)]

- renormalization of the pure scaling operator with q matrices Q:

$$\langle \mathcal{K}_\lambda \rangle \sim L^{-x_\lambda}, \quad x_\lambda = q x_1 + \Delta_\lambda \qquad x_1 = \eta^*_{(1)}, \quad \Delta_\lambda = \eta^*_\lambda.$$

- criticality in $d = 2+\epsilon$ at $t_* \simeq \epsilon/(1+6f(\gamma))$ and arbitrary γ:

$$x_\lambda \simeq \frac{\epsilon}{2(1+6f(\gamma)} \sum_{j=1}^{s} k_j(-c_j - 3\ln(1+\gamma) - k_j)$$

- symmetry relations between exponents x_λ breaks down at $\gamma \neq 0$. Examples:

$$x_{(1)} \simeq \epsilon \frac{2 - 3\ln(1+\gamma)}{2(1+6f(\gamma)} \neq x_{(2)} \simeq \epsilon \frac{1 - 3\ln(1+\gamma)}{(1+6f(\gamma)}$$

$$x_{(3)} \simeq -\epsilon \frac{9\ln(1+\gamma)}{2(1+6f(\gamma)} \neq 0.$$

Conclusions:

- we developed the theory of generalized multifractality in class C in the presence of interaction
- we constructed the pure scaling derivativeless operators in the Finkel'stein NLσM in class C
- we computed the anomalous dimensions of the pure scaling operators within two-loop approximation
- interaction breaks the exact symmetry relations between generalized multifractal exponents x_λ known in the absence of interaction

more details in Phys. Rev. B 106, 125424 (2023)

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- mapping to classical 2D percolation for some subset of pure scaling operators in class C

$$x_{(1)} = x_{(2)} = 1/4$$
$$x_{(0)} = x_{(3)} = 0$$
$$x_{(1,1)} = x_{(2,1)} = 5/4$$
$$x_{(1,1,1)} = x_{(2,1,1)} = 35/12$$
$$x_{(1,1,1,1)} = x_{(2,1,1,1)} = 21/4$$
$$x_{(1,1,1,1,1)} = x_{(2,1,1,1,1)} = 33/4$$

[Gruzberg, Ludwig, Read (1999); Beamond, Cardy, Chalker (2002); Mirlin, Evers, Mildenberger (2003); Subramaniam, Gruzberg, Ludwig (2008); Karcher, Gruzberg, Mirlin (2022)]

- gen. parabolicity does not work:
  $x_{(2)} = 1/4 \to b_C = 1/8$ but $x_{(1,1)} = 5/4 \to b_C = 5/32$!

NB it suggests absence of local conformal invariance at criticality

Reminder: pure scaling operators for non-interacting problem

- determinant pure scaling operators $\lambda = (1, \ldots, 1) \equiv (1^m)$

$$\mathcal{P}_{(1^m)}[\psi] = \det \begin{pmatrix} \psi_{1,\uparrow}(r_1) & \ldots & \psi_{m,\uparrow}(r_1) & \psi_{\bar{1},\uparrow}(r_1) & \ldots & \psi_{\bar{m},\uparrow}(r_1) \\ \\ \psi_{1,\uparrow}(r_m) & \ldots & \psi_{m,\uparrow}(r_m) & \psi_{\bar{1},\uparrow}(r_m) & \ldots & \psi_{\bar{m},\uparrow}(r_m) \\ \psi_{1,\downarrow}(r_1) & \ldots & \psi_{m,\downarrow}(r_1) & \psi_{\bar{1},\downarrow}(r_1) & \ldots & \psi_{\bar{m},\downarrow}(r_1) \\ \\ \psi_{1,\downarrow}(r_m) & \ldots & \psi_{m,\downarrow}(r_m) & \psi_{\bar{1},\downarrow}(r_m) & \ldots & \psi_{\bar{m},\downarrow}(r_m) \end{pmatrix}$$

- arbitrary pure scaling operators $\lambda = (k_1, \ldots, k_s)$

$$\mathcal{P}_\lambda[\psi] = (\mathcal{P}_{(1^1)}[\psi])^{k_1-k_2}(\mathcal{P}_{(1^2)}[\psi])^{k_2-k_3} \ldots (\mathcal{P}_{(1^{s-1})}[\psi])^{k_{s-1}-k_s}(\mathcal{P}_{(1^s)}[\psi])^{k_s}$$

[Karcher, Charles, Gruzberg, Mirlin (2021), Karcher, Gruzberg, Mirlin (2022)]