Superconducting diodes

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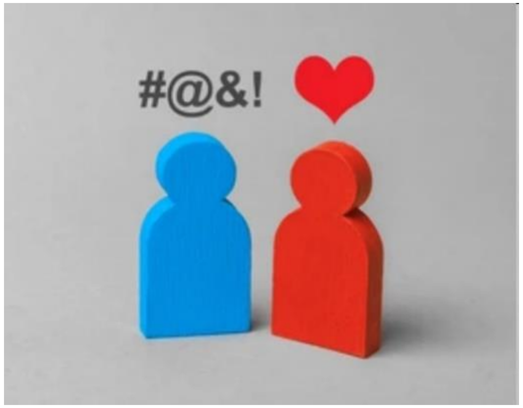
Overview

- **Introduction:** *Superconducting Diode Effect (SDE)*
- **Minimal model:** Josephson junction with finite Cooper pair momentum
- **Equilibrium properties:** Andreev bound states, anomalous Josephson effect & SDE
- **Finite bias voltage rectification: Multiple Andreev reflection (MAR) mechanism**
- **Microscopic model for SDE:** Spin-Orbit + Zeeman field
- **Conclusions & Outlook**

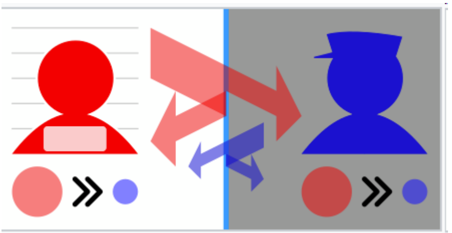
Zazunov, Rech, Jonckheere, Grémaud, Martin & Egger,
arXiv:2307.14698 & arXiv:2307.15386

Diode: nonreciprocal/asymmetric response



Nonreciprocal
feelings

asymmetric response (sports/politics)



one-way (semi-transparent) mirror

Semiconductor diode

Nonreciprocal DC current vs. applied voltage:

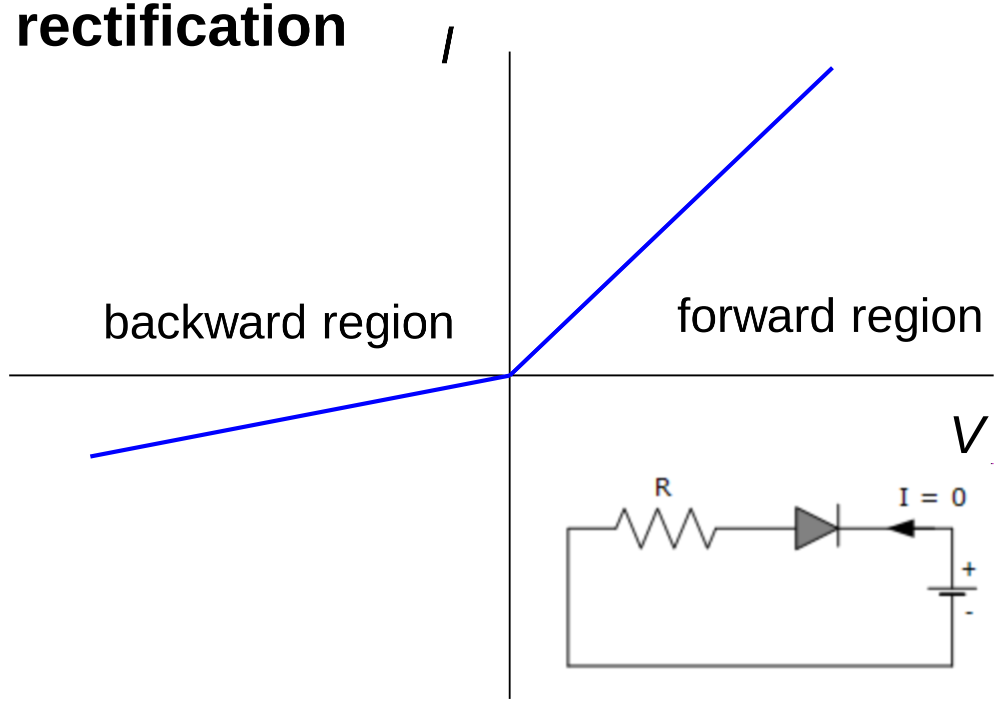
$$V = R(I)I \qquad R(-I) \neq R(I)$$

resistive (dissipative) rectification

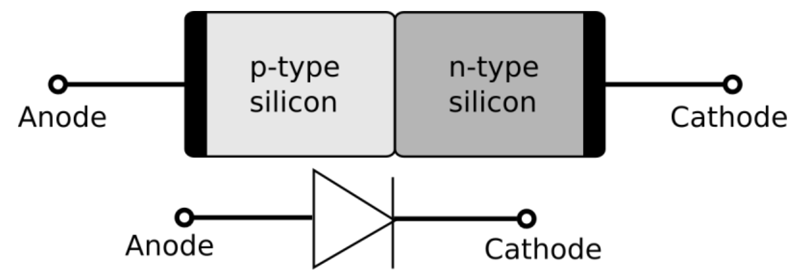
Ideal diode behavior:

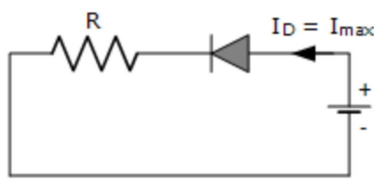
$$R(I) = \begin{cases} R_0 & I > 0 \\ \infty & I < 0 \end{cases}$$

***p-n* junction diode**

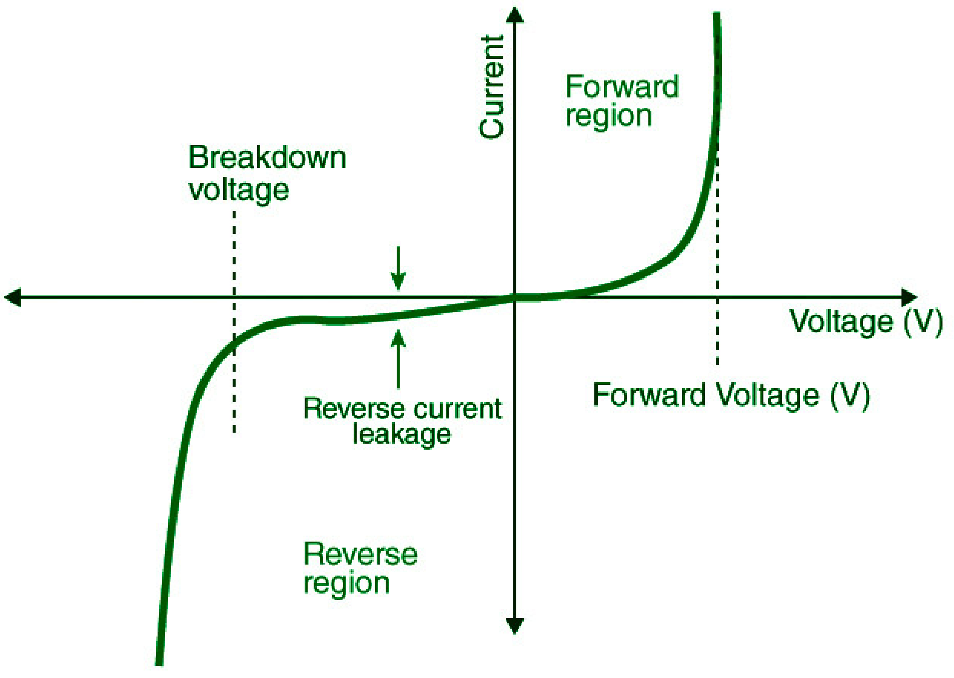


p-n junction diode





Nonreciprocity –
why does it work?



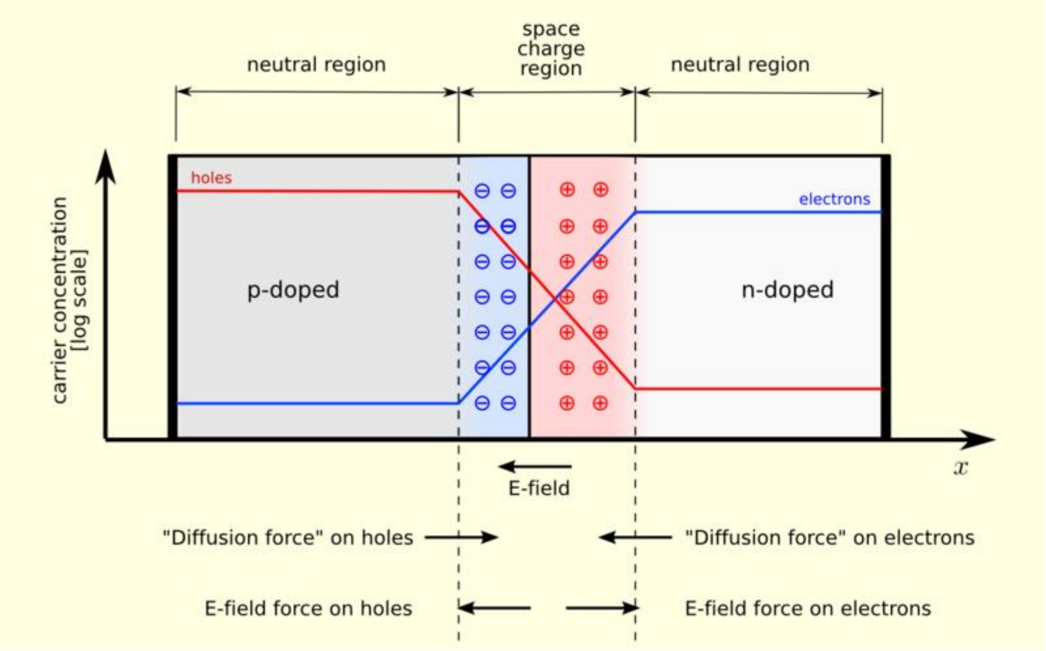
Inversion symmetry breaking

Key requirement for nonreciprocal charge transport: broken inversion symmetry

→ current direction changes depletion layer thickness

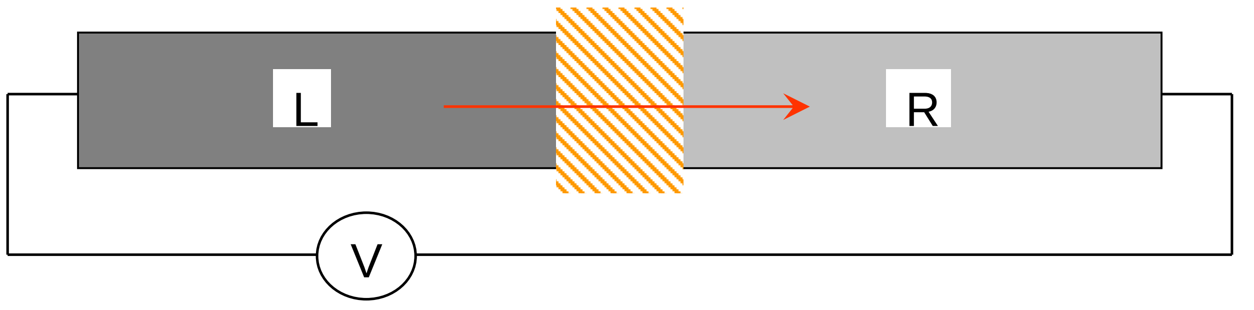
→ **asymmetric resistivity**

Inversion symmetry breaking:
necessary but not sufficient criterion



Broken inversion symmetry does **not** guarantee directional response…

Example: tunneling current in conventional (N-I-N) junction



1D scattering problem with asymmetric potential barrier

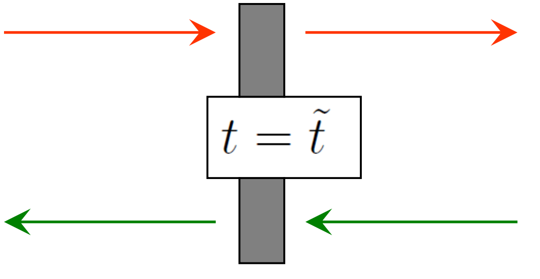
S-matrix is symmetric in the presence of time-reversal symmetry

$$S = \begin{pmatrix} r & \tilde{t} \\ t & \tilde{r} \end{pmatrix} = S^T$$

$$I(V) \propto \int d\epsilon \nu_L(\epsilon)\nu_R(\epsilon)\left[n_F(\epsilon + V) - n_F(\epsilon)\right]$$

$$\rightarrow \; I(-V) = -I(V)$$

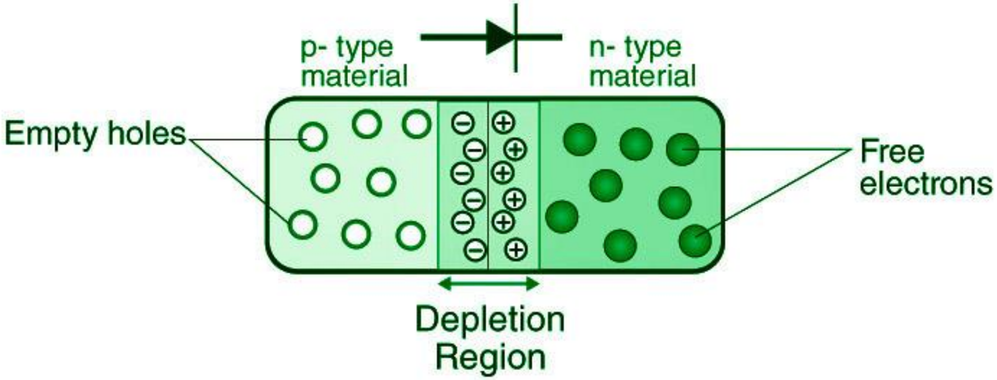
no rectification!



- Nonreciprocal current response often found if both inversion and time reversal symmetry (TRS) are broken (e.g., magnetic field)

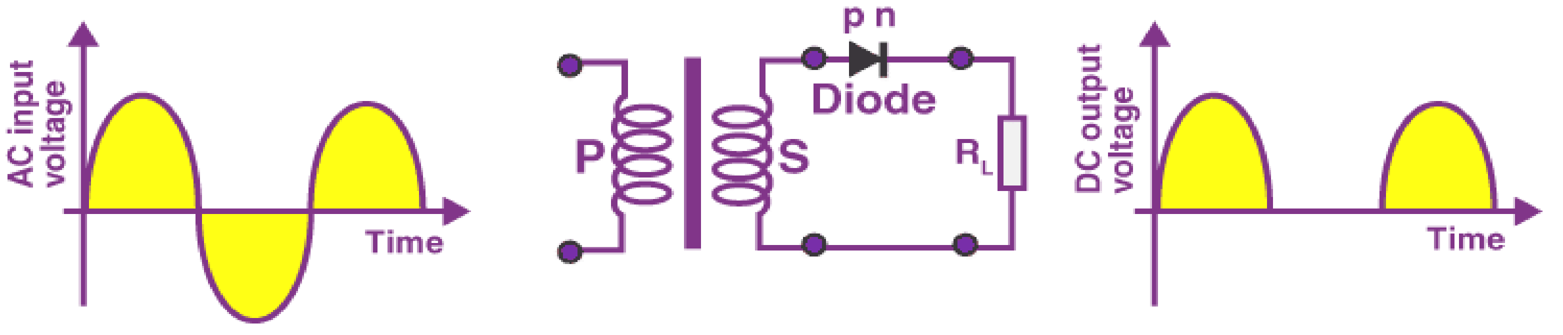
- can also be achieved through interaction effects in TR-invariant systems

p-n junction diode:
TRS not broken in equilibrium $(V = 0)$



Rectifiers are usually based on semiconductor diodes: convert alternating EM field into directed current

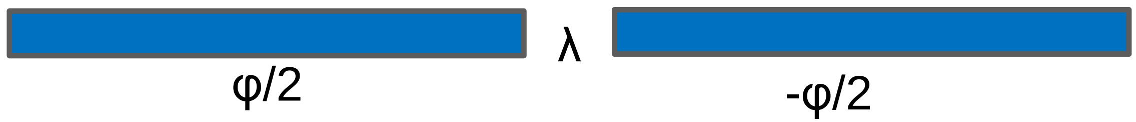
(assume ideal diode behavior)



Q1: Superconducting dissipationless (V=0) version of classical diode ? → Superconducting Diode Effect (SDE)

Q2: Superconducting dissipative version (finite V) ?
→ SDE out of equilibrium

Brief reminder: Josephson effect



- **Tunnel contact** (tunnel amplitude λ) between two (*s*-wave BCS) superconductors at phase difference φ
- **Tunneling of Cooper pairs (charge $2e$)** $\rightarrow$

  2π-periodic coupling energy $\quad E_{Jos}(\varphi) = -E_J \cos(\varphi)$

  with Josephson coupling $E_J \sim \lambda^2$
- Josephson equilibrium (DC) current-phase relation (CPR) from phase derivative of free energy:

$$I(\varphi) = \frac{2e}{\hbar}\frac{dE_{Jos}(\varphi)}{d\varphi} = I_c \sin\varphi$$ with critical current $I_c = \frac{2eE_J}{\hbar}$

dissipationless supercurrent

AC Josephson effect

- Application of constant bias voltage V → time-dependence of phase difference according to **second Josephson relation**

$$\varphi(t) = \varphi(0) + \frac{2e}{\hbar} V t$$

- **NB:** First Josephson relation holds only in tunneling limit! In general non-sinusoidal CPR.
  - Single-channel Josephson junction with transmission probability τ: $$I(\varphi) = \frac{e\Delta}{2\hbar} \frac{\tau \sin\varphi}{\sqrt{1-\tau \sin^2\frac{\varphi}{2}}}$$

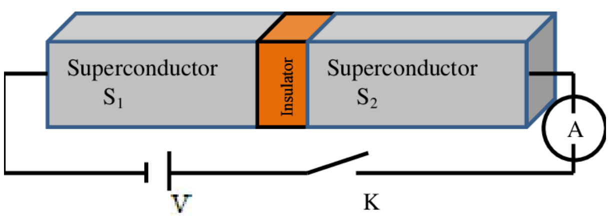
Superconducting diode effect (SDE)

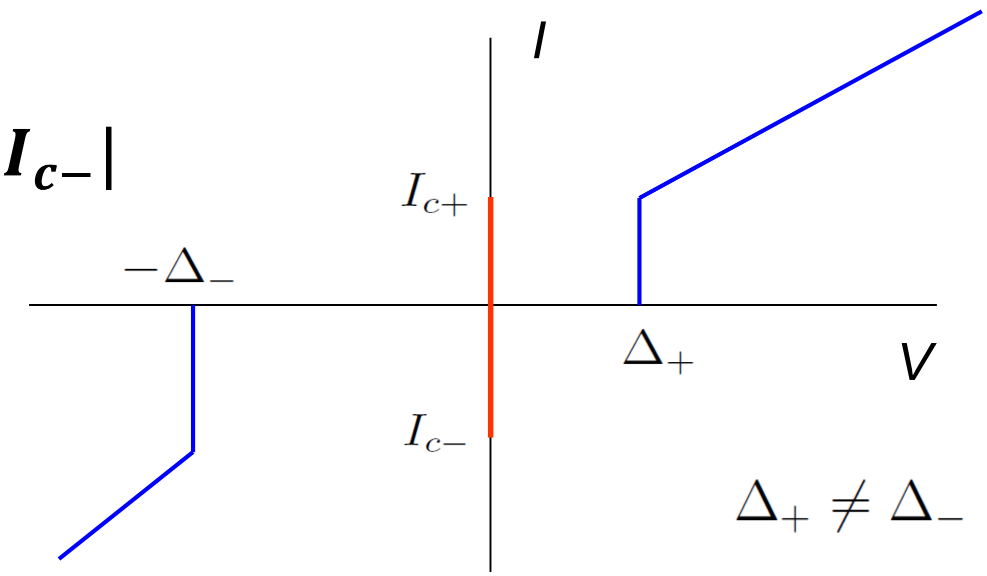
SDE: dissipationless equilibrium supercurrent flows along one direction but not in the opposite one

→ need polarity-dependent critical current $I_{c+} \neq |I_{c-}|$

Rectification regime:

$$I_{c+} < |I| < |I_{c-}|$$





NB: Here for Josephson junction between „skewed“ superconductors (SCs), but SDE exists also for junction-free bulk SCs…

SDE efficiency

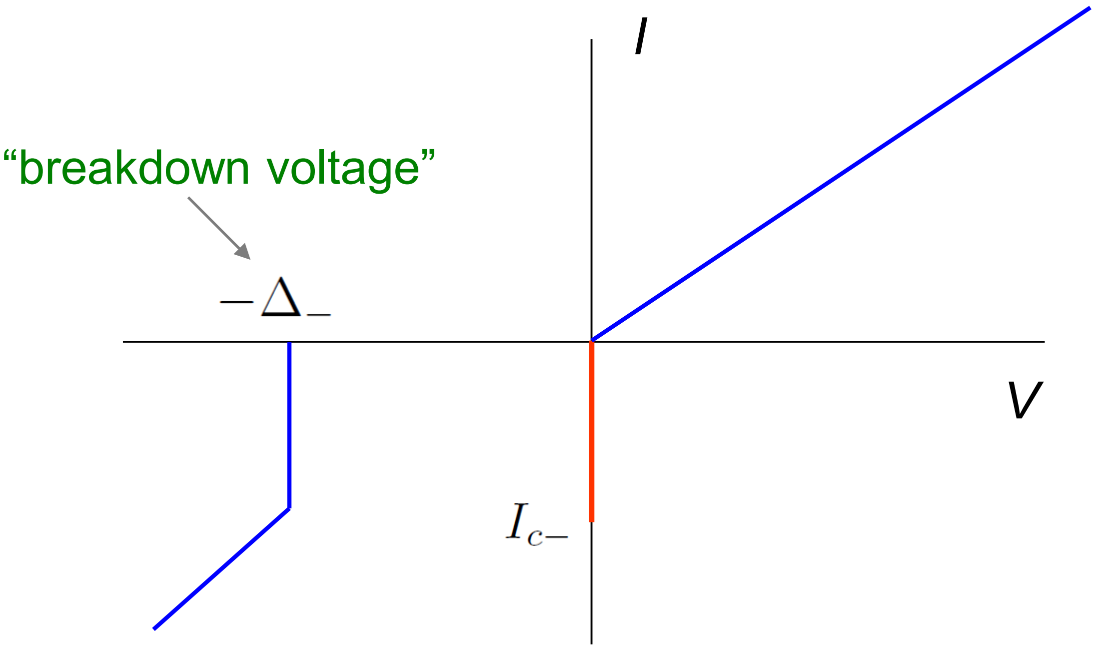
SDE efficiency

$$\eta_0 = \left| \frac{I_{c+} - |I_{c-}|}{I_{c+} + |I_{c-}|} \right|$$

Ideal case: $\eta_0 = 1$

$$I_{c+} = 0$$

$$\Delta_+ = 0$$



First report of SDE (in bulk SC)

nature

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nature > articles > article

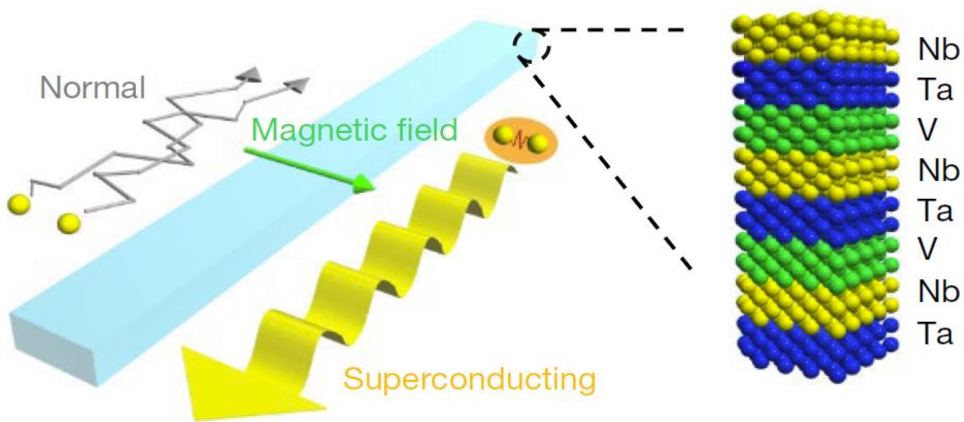
Article | Published: 19 August 2020

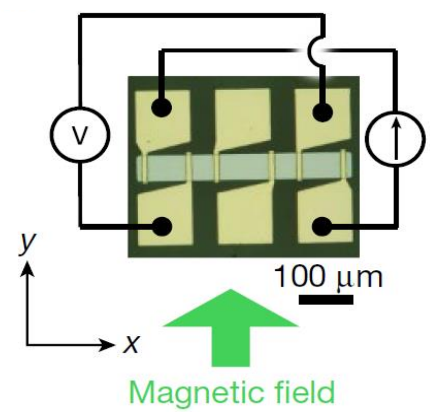
Observation of superconducting diode effect

Fuyuki Ando, Yuta Miyasaka, Tian Li, Jun Ishizuka, Tomonori Arakawa, Yoichi Shiota, Takahiro Moriyama, Youichi Yanase & Teruo Ono

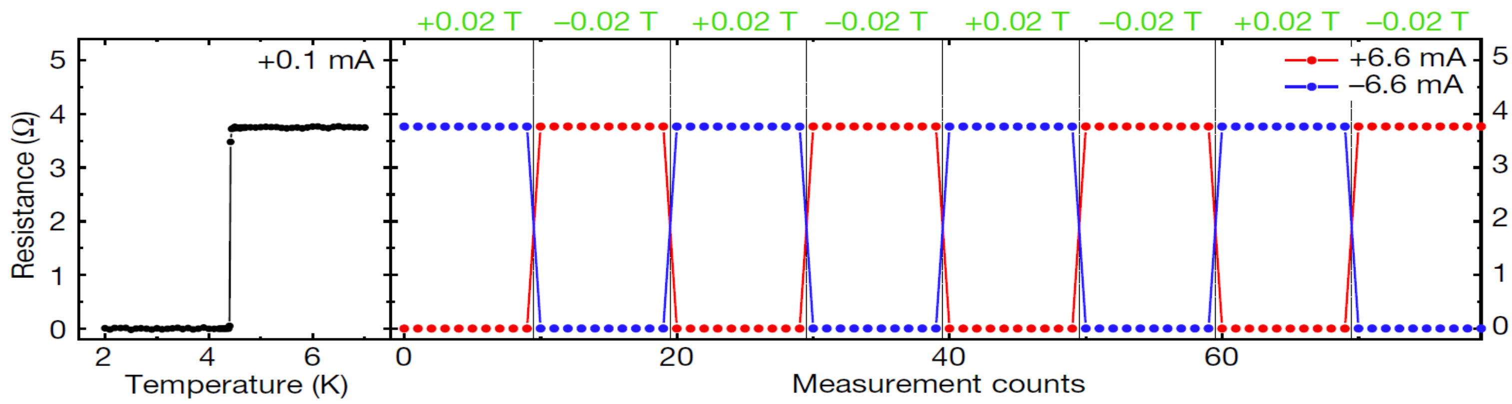
Nature **584**, 373–376 (2020) | Cite this article

Rashba superconductor: noncentrosymmetric artificial superlattice with [Nb/V/Ta] units

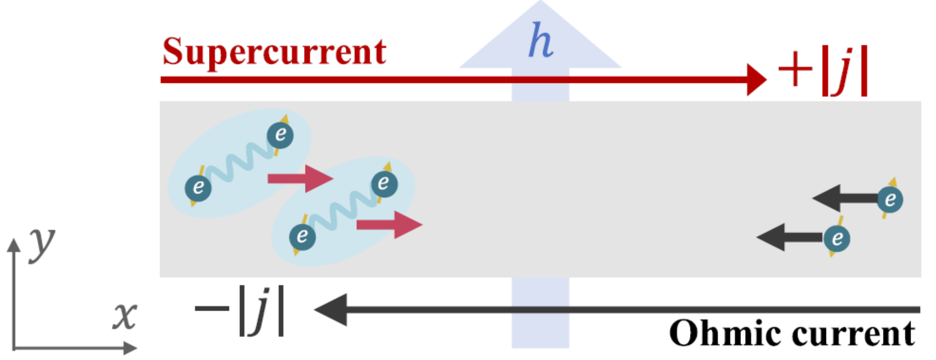




Demonstration of magnetically controllable **superconducting diode**: alternating switching between super- and normal-conducting state by **changing the sign of the applied current or a small magnetic field**



Superconducting diode effect:
nonreciprocity of critical current for metal-superconductor transition

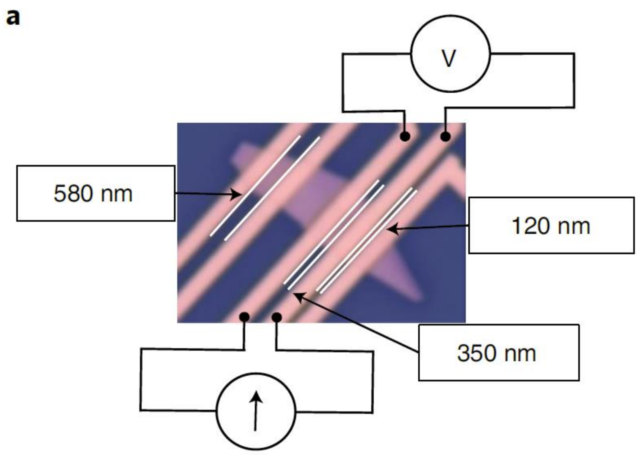


SDE

many experimental confirmations since 2020

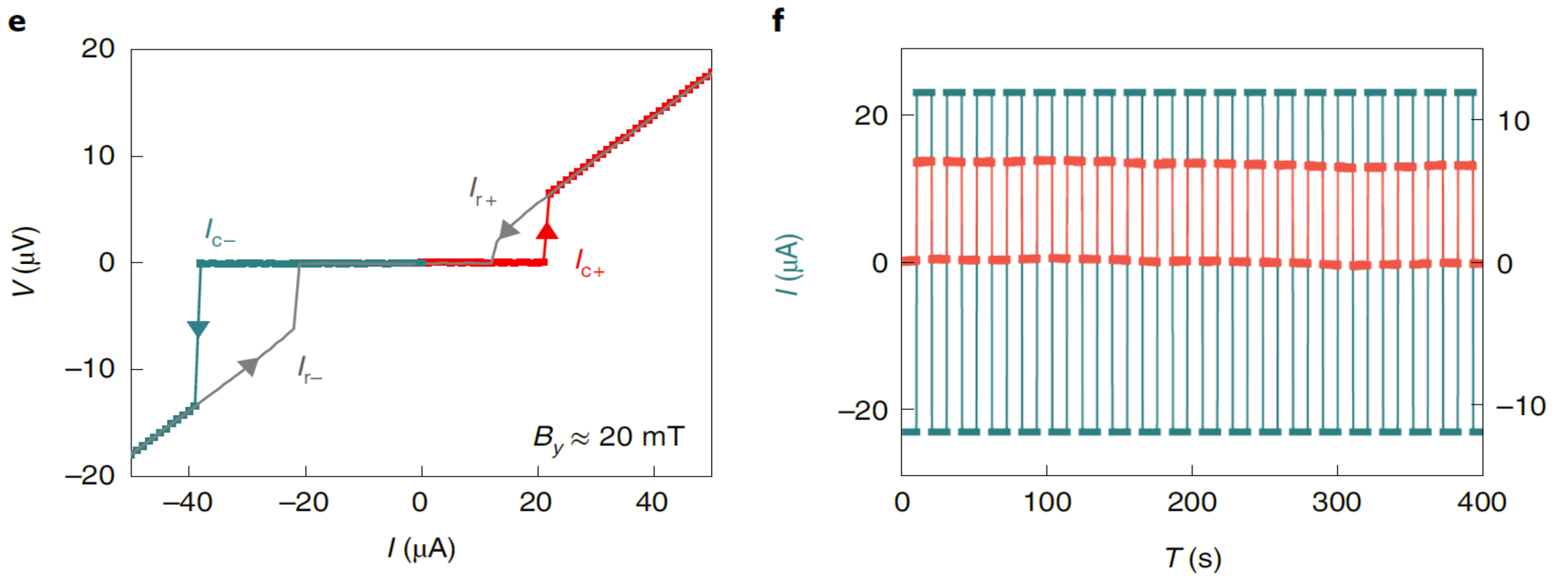
Ando et al., Nature 2020

→ ***very hot field***

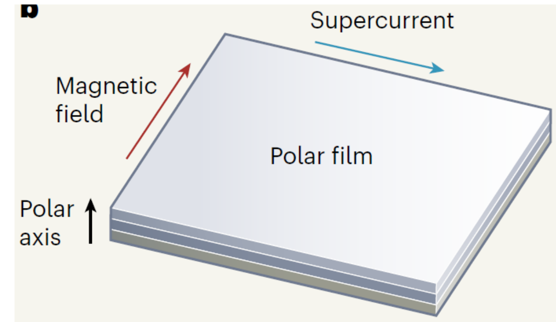


Example: Rectification in NiTe2 JJ

Pal et al., Nat. Phys. 2022



Theory: Bulk case



J. Phys.: Condens. Matter **8** (1996) 339–349. Printed in the UK

The Ginzburg–Landau equation for superconductors of polar symmetry

Victor M Edelstein
Institute of Solid State Physics, Russian Academy of Science, Chernogolovka, Moscow region 142432, Russia

GL free energy in the presence of Rashba spin orbit term:

$$H_{so} = \frac{\alpha}{\hbar}(\boldsymbol{p} \times \boldsymbol{c}) \cdot \boldsymbol{\sigma}$$

$$\Omega(\psi, \psi^*) = \int \mathrm{d}^3 r \left[\frac{1}{\eta} \left(\frac{T - T_c}{T_c} |\psi|^2 + \frac{1}{2n} |\psi|^4 \right) + \frac{1}{4m} (\boldsymbol{\Pi}^* \psi^*) \cdot (\boldsymbol{\Pi} \psi) - \frac{1}{2} \kappa (\boldsymbol{c} \times \boldsymbol{B}) \cdot (\psi^* \boldsymbol{\Pi} \psi + \psi \boldsymbol{\Pi}^* \psi^*) \right]$$

$$\boldsymbol{\Pi} = -\mathrm{i}\boldsymbol{\nabla} - (2e/c)\boldsymbol{A}$$

Critical current different for opposite directions:

$$J_c(\boldsymbol{B}) = J_c(0) \left[1 + (\boldsymbol{c} \times \boldsymbol{B}) \cdot \hat{\boldsymbol{J}} f_3 \, \delta \frac{3(7\zeta(3))^{1/2}}{8 H_{c2} p_F \xi(T)} \right]$$

Finite Cooper pair momentum (FCPM)

Supercurrent-carrying state = macroscopic condensate of Cooper pairs at finite momentum Q

- Gauge $A = 0 \rightarrow$ GL order parameter $\Psi_Q(x) = \Psi e^{iQx/\hbar}$ oscillates in space!

$\rightarrow$ Superconductor (SC) condensate energy $E_Q = \frac{NQ^2}{2(2m)}$

$\rightarrow$ energy cost with growing Q limits max. supercurrent

- Galilei-invariant system: current $J_Q = e\partial_Q E = Nev_Q$ with superfluid velocity $v_Q = Q/2m$
- $Q \neq 0$ state is metastable (very long lifetime), carries non-dissipative supercurrent

(very different from current-carrying state in a normal metal!)

Finite Cooper pair momentum: SDE

Ground state with $Q \neq 0$ is (almost) sufficient (but not necessary) for obtaining the SDE !

→ **broken TRS and broken inversion symmetry** ✓

Recipes to realize such a ground state:

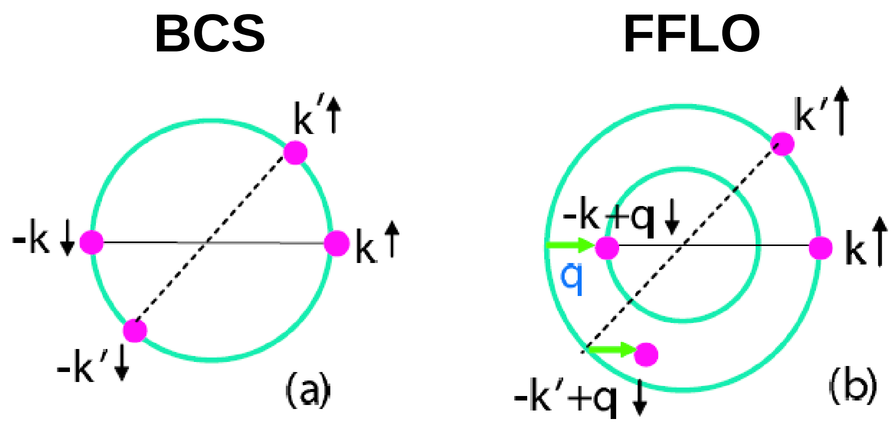
- **Fulde-Ferrell (FF) state:** spontaneous symmetry breaking in Zeeman field for clean SC [Larkin-Ovchinnikov (LO) state combines degenerate $\pm Q$ states & preserves TRS] *Daido et al. PRL 2022*
- **Helical SC:** interplay of spin-orbit coupling and Zeeman field generates effective $Q \neq 0$ ***(magnetochiral anisotropy),*** **robust against disorder** *Edelstein JPCM 1996; Zazunov, Egger, Jonckheere & Martin, PRL 2009; Yuan & Fu, PNAS 2022*
- **SC-ferromagnet heterostructure:** leakage of magnetism induces FF state in SC *Mironov, Mel'nikov & Buzdin, APL 2018*

FFLO superconductor

Review: Matsuda & Shimahari, JPSJ 2007

$$H = H_{\rm BCS} + \mu_B H \sum_i \sigma_{z,i}$$

spatially uniform magnetic Zeeman field → FFLO pairing with FCPM q advantageous → spatial modulation of order parameter



$$\Delta_{FF}(\mathbf{r}) = \Delta e^{i\mathbf{qr}}$$

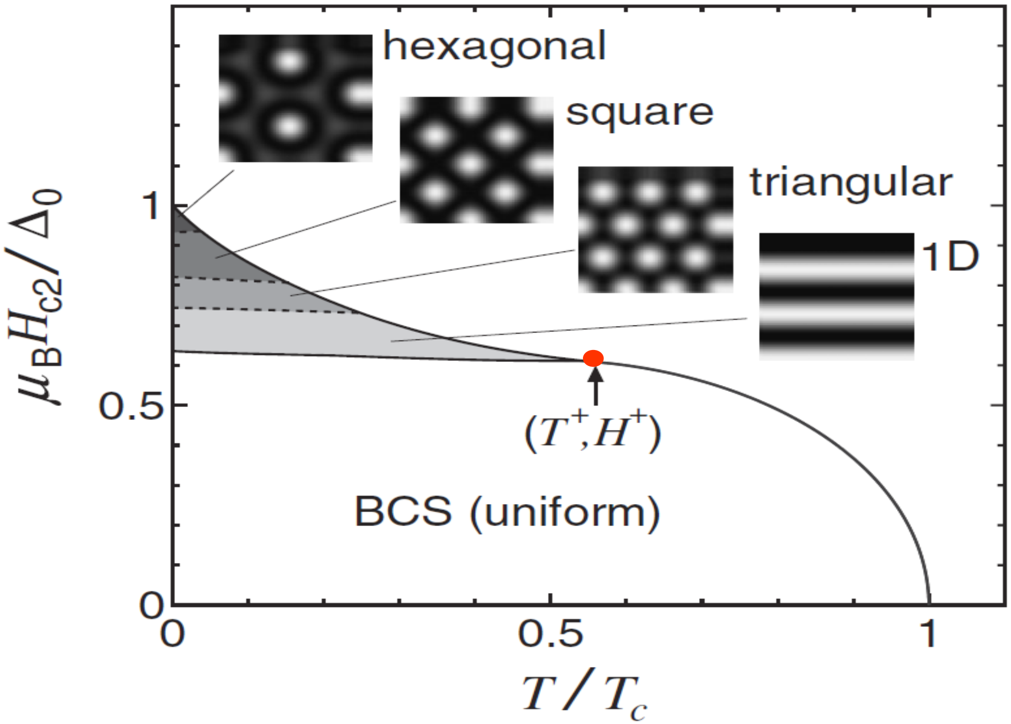
But: one needs clean limit and absence of orbital field effects

LO: In gap equation, **q** and **-q** solutions are **degenerate** states. Degeneracy is lifted by linear combinations of **q** and **-q**

$$\Delta_{LO}(\mathbf{r}) = 2\Delta\cos(\mathbf{qr})$$

Phase diagram of 2D superconductor
in parallel magnetic field
(no orbital effects)

Normal (Δ=0), BCS (uniform) & FFLO phase meet at tricritical point



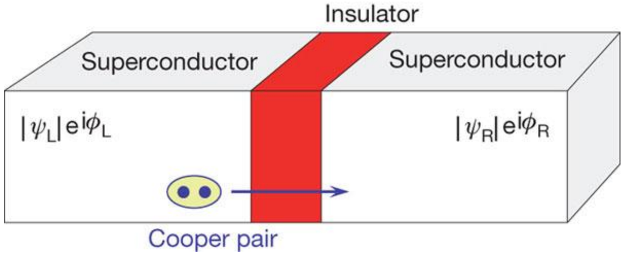
From now on: consider Josephson diode = SDE in a Josephson junction

Anomalous Josephson effect

In presence of **either** TRS **or** inversion symmetry: CPR **odd** under phase reversal

$$I(\varphi) = -I(-\varphi)$$
$$I(0) = 0$$

$$I(\varphi) = \sum_{n=1}^{\infty} b_n \sin(n\varphi)$$



$$\varphi = \phi_L - \phi_R$$

If both symmetries are broken, e.g., by FCPM mechanism, one may find the **anomalous Josephson effect:**
finite supercurrent flows for $\varphi = 0$ $\quad I(\varphi) \neq -I(-\varphi)$

Tunnel junction limit → standard Josephson relation but with **anomalous phase shift:** $I(\varphi) = I_c \sin(\varphi + \varphi_0)$ → no SDE
→ **SDE only possible beyond tunnel junction limit!**

Josephson diode conditions

Keeping only the **lowest few harmonics in CPR:**

$$I(\varphi) \approx b_1 \sin\varphi + a_1 \cos\varphi + b_2 \sin(2\varphi)$$

generated by interplay of **spin-orbit coupling & magnetic Zeeman field** (→ anomalous Josephson effect)

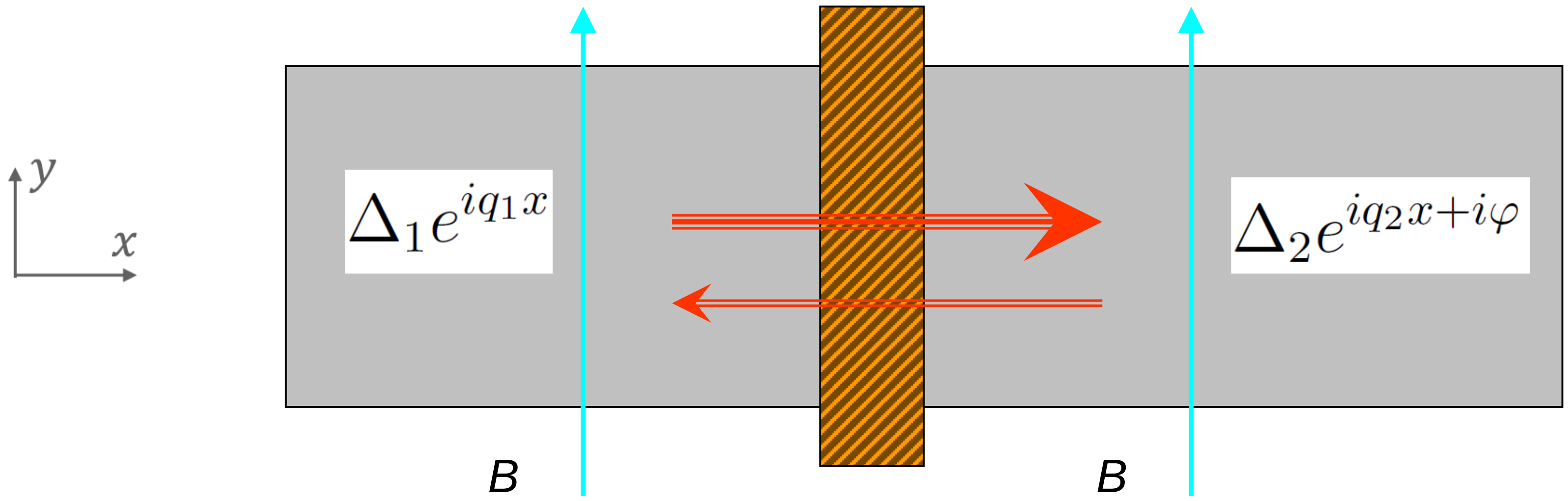
skewed CPR from **CP cotunneling** (→ effect beyond tunneling limit)

From this CPR → $\boldsymbol{\Delta I_c = I_{c+} - |I_{c-}| \propto a_1 b_2}$

→ SDE generically happens away from deep tunneling limit if anomalous Josephson effect is present

Zazunov, Egger, Jonckheere & Martin, PRL 2009; Brunetti, Zazunov, Kundu & Egger, PRB 2013

Josephson diode effect



Josephson junction geometry with helical SCs

Case study**: SDE due to FCPM**

FCPM = finite Cooper pair momentum

nature physics

ARTICLES

https://doi.org/10.1038/s41567-022-01699-5

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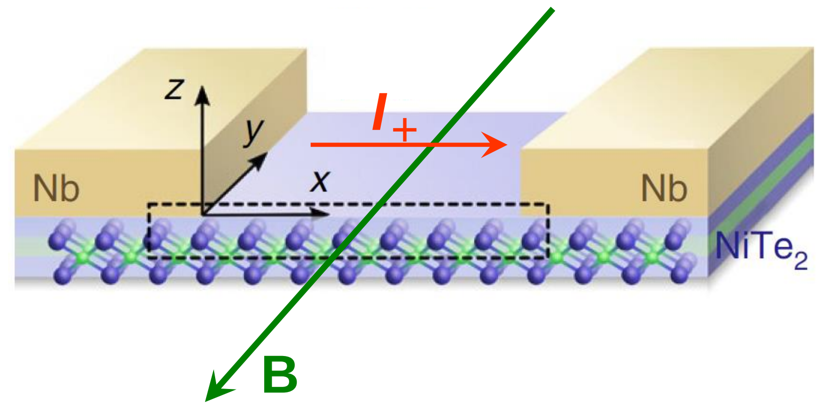
OPEN

Josephson diode effect from Cooper pair momentum in a topological semimetal

Banabir Pal[1,4], Anirban Chakraborty[1,4], Pranava K. Sivakumar[1,4], Margarita Davydova[2,4], Ajesh K. Gopi[1], Avanindra K. Pandeya[1], Jonas A. Krieger[1], Yang Zhang[2], Mihir Date[1], Sailong Ju[3], Noah Yuan[2], Niels B. M. Schröter[1], Liang Fu[2] ✉ and Stuart S. P. Parkin[1] ✉

- *large Josephson diode effect observed*
- *FCPM model seemingly explains the main observations*



Measured SDE efficiency of up to $\eta_0 \approx 40\%$

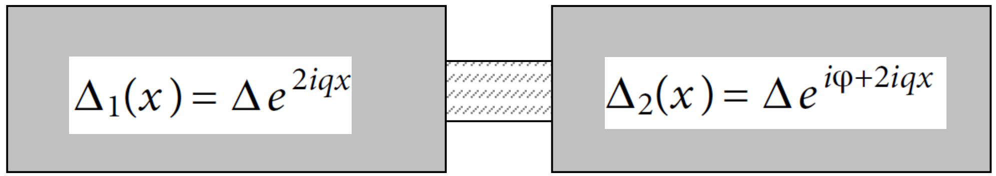
$$\eta_0 = \left|\frac{I_{c+} - |I_{c-}|}{I_{c+} + |I_{c-}|}\right|$$

FCPM model for Josephson diode effect

Davydova, Prembabu & Fu, Sci. Adv. (2022)

Josephson junction as **short weak link** between **helical SCs**

FCPM $Q = 2q$ (assumed identical on both sides)



- **FCPM** captures simultaneous breaking of TR & inversion symmetries
- **Single-channel limit:** Transmission probability τ away from deep tunneling regime ($\tau \ll 1$)

$\rightarrow$ 1D low-energy **Bogoliubov-de Gennes (BdG) Hamiltonian** with weak link at $x = 0$ and coherence length $\xi = \hbar v_F/\Delta$

$$H = \int dx\, \Psi^\dagger \begin{pmatrix} \xi(-i\partial_x) & \Delta(x) \\ \Delta^*(x) & -\xi(-i\partial_x) \end{pmatrix} \Psi, \quad \Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^\dagger \end{pmatrix}$$

Nambu spinor states at x=0- and x=0+ connected by τ-dependent transfer matrix: **matching condition**

Ballistic limit: full transparency $\tau = 1$

No backscattering → two independent **chiral channels**
(right/left movers $\alpha = \pm$)

Matching condition: $\Psi(0^-) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \Psi(0^+)$

- kinetic energy linearized near Fermi points
- q and φ dependent SC phases gauged away from $\Delta(x)$ →

$$H_{\alpha=\pm} = \begin{pmatrix} \alpha v_F(-i\partial_x + q) & \Delta \\ \Delta & -\alpha v_F(-i\partial_x - q) \end{pmatrix}$$

Doppler shift from FCPM

φ = 0 (1D bulk SC): quasiparticle dispersion has **different spectral gaps for right and left movers**

$$\Delta_\pm = \Delta \pm v_F|q|$$

Here: we assume $v_F|q| < \Delta$, otherwise gapless SC

Spectral regimes

Depending on quasiparticle energy E, different types of quasiparticle states exist (for arbitrary τ):

$|\boldsymbol{E}| < \boldsymbol{\Delta}_-$: **current-carrying Andreev bound states** (subgap states spatially localized near x=0)

$|\boldsymbol{E}| > \boldsymbol{\Delta}_+$: propagating **continuum quasiparticles**

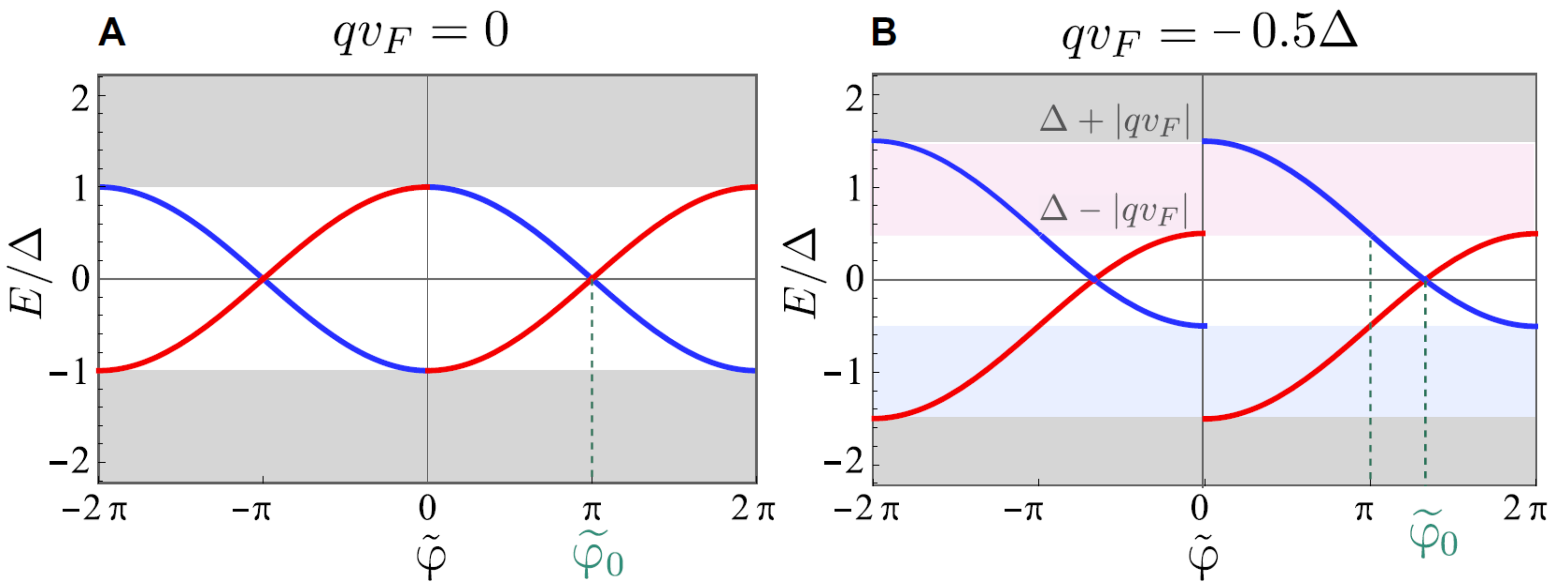
$\boldsymbol{\Delta}_- < |\boldsymbol{E}| < \boldsymbol{\Delta}_+$: **mixed-character states** (evanescent in one direction, propagating in the other)

Eigenstates in different regimes are related by analytic continuation → unified approach

Andreev bound states: ballistic case

Closed analytical results possible in ballistic case (even for V>0)

$$E_{\pm}(\varphi) = \pm\Delta(\cos(\varphi/2) - q\xi)$$



Josephson diode effect: ballistic limit

CPR = Andreev + continuum contribution, at T=0 given by

$$I_A(\varphi) = \frac{e\Delta}{\hbar}\sin\left(\frac{\varphi}{2}\right)\ \mathrm{sgn}\left(\cos\left(\frac{\varphi}{2}\right) - q\xi\right)$$

$$I_{cont} = \frac{2e\Delta}{\pi\hbar}\, q\xi$$

→ polarity-dependent critical currents

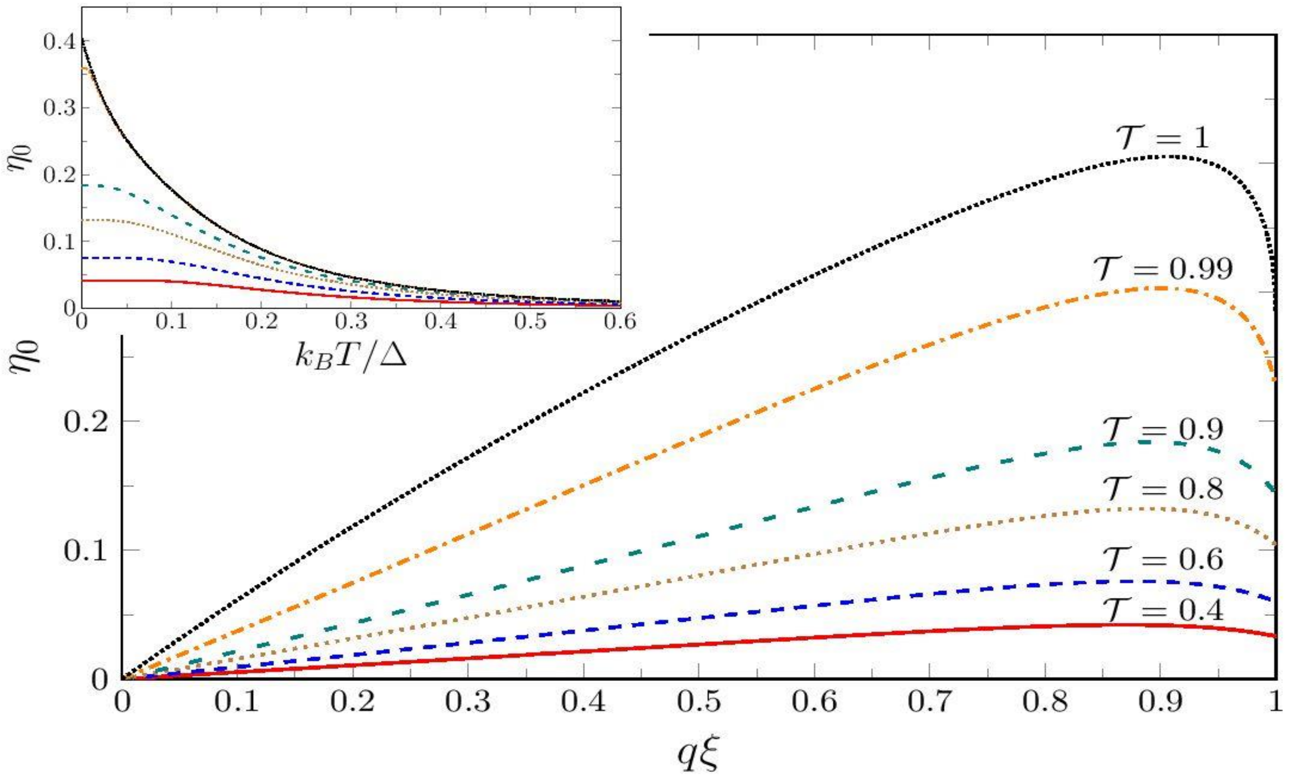
→ SDE efficiency follows as

$$\eta_0 = 1 - \frac{2 - 4q\xi/\pi}{1 + \sqrt{1 - (q\xi)^2}}$$

→ maximal SDE efficiency is $\eta_0 \approx 40\%$, reached for $q\xi \approx 0.9$

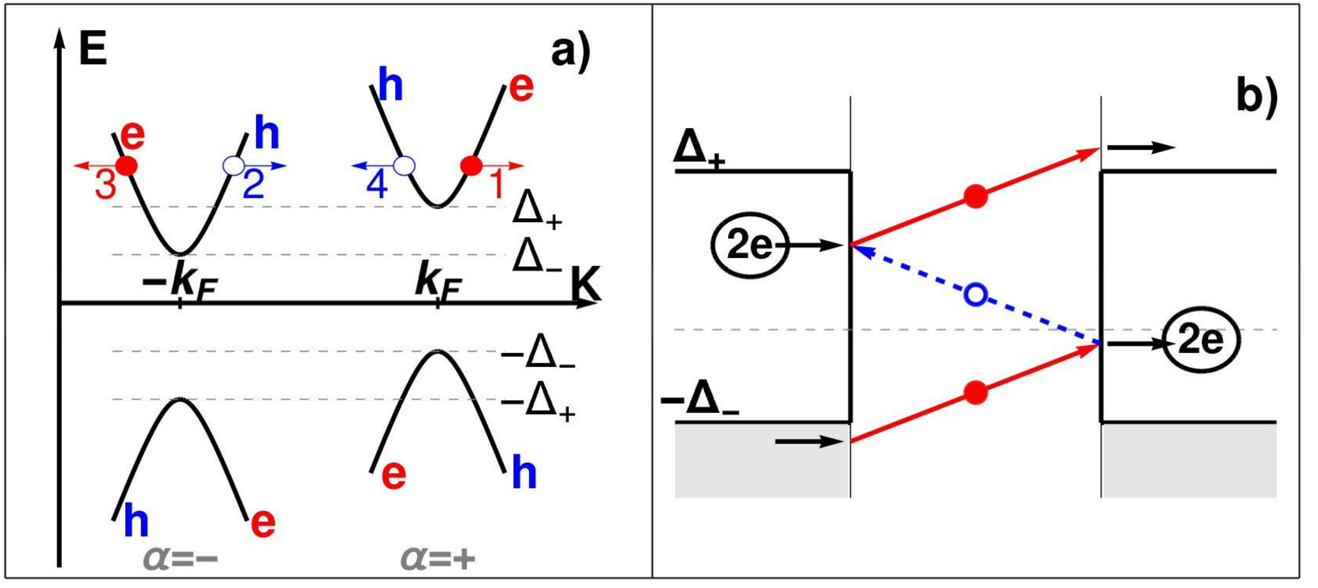
$$\eta_0 = \frac{I_{c+} - |I_{c-}|}{I_{c+} + |I_{c-}|}$$

SDE efficiency for arbitrary τ



- Rapid decrease of SDE efficiency for poor junction transmission
- Optimal working point $q\xi \approx 0.9$ approximately independent of τ
- Inset: thermal degradation at $q\xi = 0.9$

Nonequilibrium Josephson diode: dispersion & scattering states



- **Four types of incoming quasiparticle states:** electron- or hole-like states, incoming from left or right side
- **Scattering processes at junction: Andreev reflection vs normal reflection** (ballistic case: only AR)
- With voltage: **Multiple Andreev reflection (MAR) ladder**

Andreev reflection amplitude

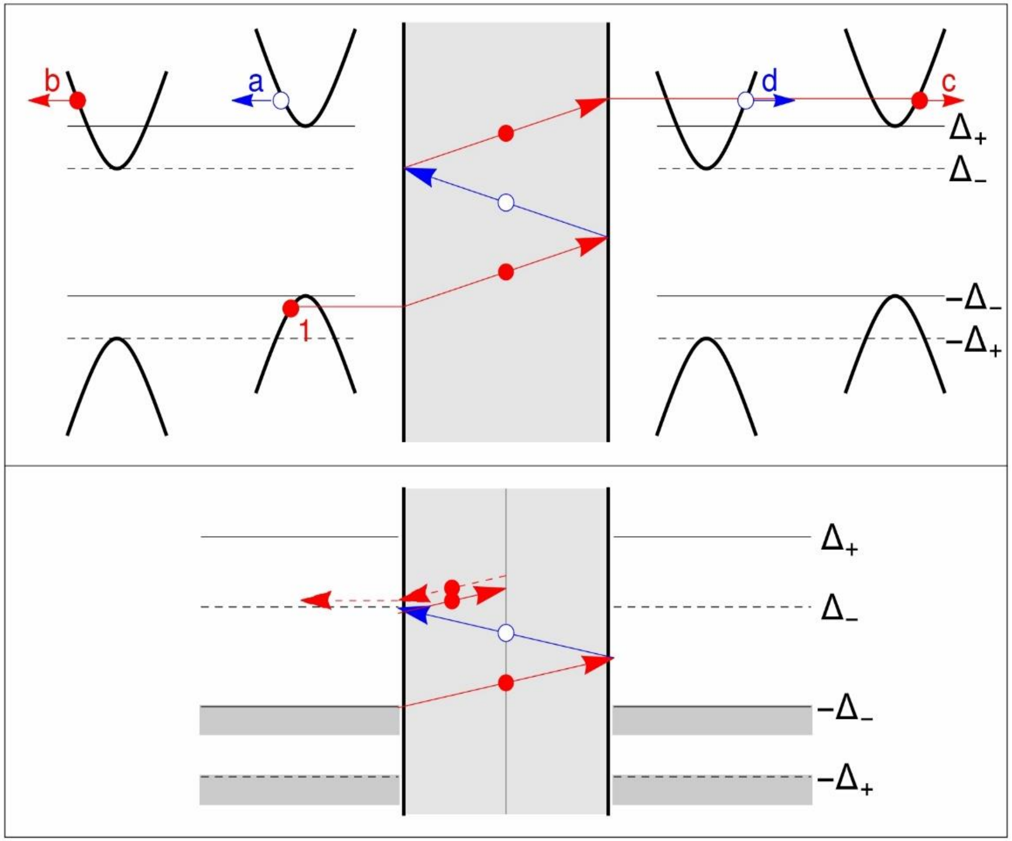
Amplitude for Andreev reflection at NS interface for incoming state with energy E:

$$\rho(E) = \begin{cases} e^{-i\cos^{-1}\left(\frac{E}{\Delta}\right)}, & |E| < \Delta, \\ \mathrm{sgn}(E)\,\dfrac{|E| - \sqrt{E^2 - \Delta^2}}{\Delta}, & |E| > \Delta \end{cases}$$

- Subgap energies: complex phase factor, $|\rho(E)| = 1$
- Above-gap energies: real number with $|\rho(E)| < 1$

Martin-Rodero and Levy Yeyati, Adv. Phys. 2011

Scattering states



Incoming state of type 1, incident energy E

Outgoing state:
scattering amplitudes
(a_n, b_n, c_n, d_n)
at energy $E_n = E + neV$

Example for MAR process with one normal reflection event (for $\tau < 1$)

NB: outgoing-state energy E_n can be in any of the three spectral regimes!

NS junction: nonlinear conductance

- Warm-up exercise: contact between normal metal (N) and helical SC (S) with transparency τ
  - normal metal modeled as Δ=0 limit of SC
  - **Conductance** $G(V) = \frac{dI}{dV}$ from solution of scattering problem & Landauer-Büttiker formula
- Ballistic case (T=0):

$$\frac{G}{2e^2/h} = 1 + \frac{1}{2}\sum_{\alpha=\pm}\left(\Theta(1-|\nu_\alpha|) + \frac{\Theta(|\nu_\alpha|-1)}{\left(|\nu_\alpha| + \sqrt{\nu_\alpha^2 - 1}\right)^2}\right)$$

with $\nu_\alpha = \frac{eV}{\Delta} - \alpha q \xi$

NS junction conductance

Ballistic conductance obeys

$$\frac{2e^2}{h} \leq G(V) \leq \frac{4e^2}{h}$$

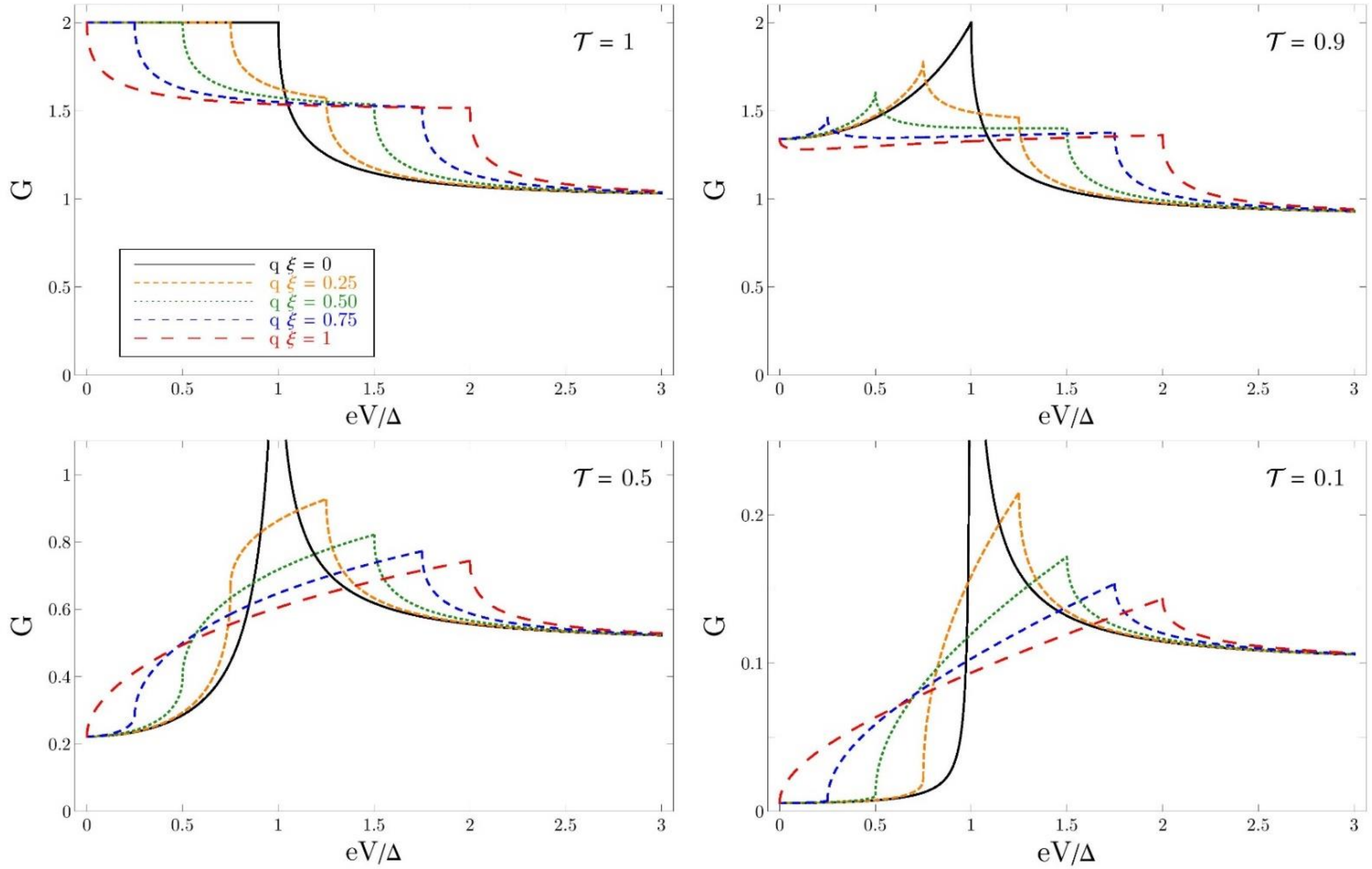
Two spectral gaps clearly visible in kink-like features

- Subgap regime $eV < \Delta_-$: perfect Andreev reflection

$$\rightarrow G = \frac{4e^2}{h}$$

Away from ballistic regime: numerical solution

NS junction conductance



(G in $\frac{2e^2}{h}$)

- **FCPM → two kink-like features, nonuniversal peak height**
- NS conductance is symmetric in V → **no rectification**

Transport through Josephson diode

- Matching conditions imply **recurrence relations** for scattering amplitudes: Multiple Andreev Reflections (MAR)
- Symmetry relations → restrict incoming states to type s=1,2
- Amplitudes (a_n, b_n) can be eliminated →

$$X_{n,n+1}(r)\begin{pmatrix} c_{n+1} \\ d_{n+1} \end{pmatrix} = X^{-1}_{n-1,n}(-r)\begin{pmatrix} c_{n-1} \\ d_{n-1} \end{pmatrix} + \delta_{n,0}\begin{pmatrix} \delta_{s,1}\, J_+ \\ \delta_{s,2}\, J_- \end{pmatrix}$$

$$X_{n,n+1}(r) = \frac{1}{\sqrt{\tau}}\begin{pmatrix} \rho_{+,n} & r\rho^{-1}_{+,n}\rho_{-,n+1} \\ r & \rho_{-,n+1} \end{pmatrix} \quad \text{with} \quad r = \sqrt{1-\tau},$$

$$\rho_{\alpha=\pm,n} = \rho(E - \alpha v_F q + neV),$$

$$J_\alpha = \alpha(\rho^{-1}_{\alpha,0} - \rho_{\alpha,0})\sqrt{\frac{2}{1+\rho^2_{\alpha,0}}}$$

Josephson diode: ballistic limit

Analytical solution of recurrence relations for $\tau = 1$ gives

$$I_q(V) = I_{q=0}(V) + \frac{2e\Delta}{\pi\hbar}\, q\xi$$

Analytical solution for ballistic limit in standard case *q=0*

Current carried by Cooper pairs with FCPM

Averin & Bardas, PRL 1995

Limits:

$\boldsymbol{eV \gg \Delta}$: effectively normal-conducting contact, $I_0(V) = \frac{2e^2}{h} V$

$\boldsymbol{eV \ll \Delta}$: time-averaged Andreev level current, $I_0(V) = \frac{4e\Delta}{h}\,\mathrm{sgn}(V)$

For finite reflection amplitude ($\tau < 1$):
Numerical analysis of recurrence relations…

Nonlinear conductance of Josephson diode

Numerical results (*G* in units of $2e^2/h$)

$$\tau = 0.7,\ q\xi = 0.3,\ \ T = 0$$

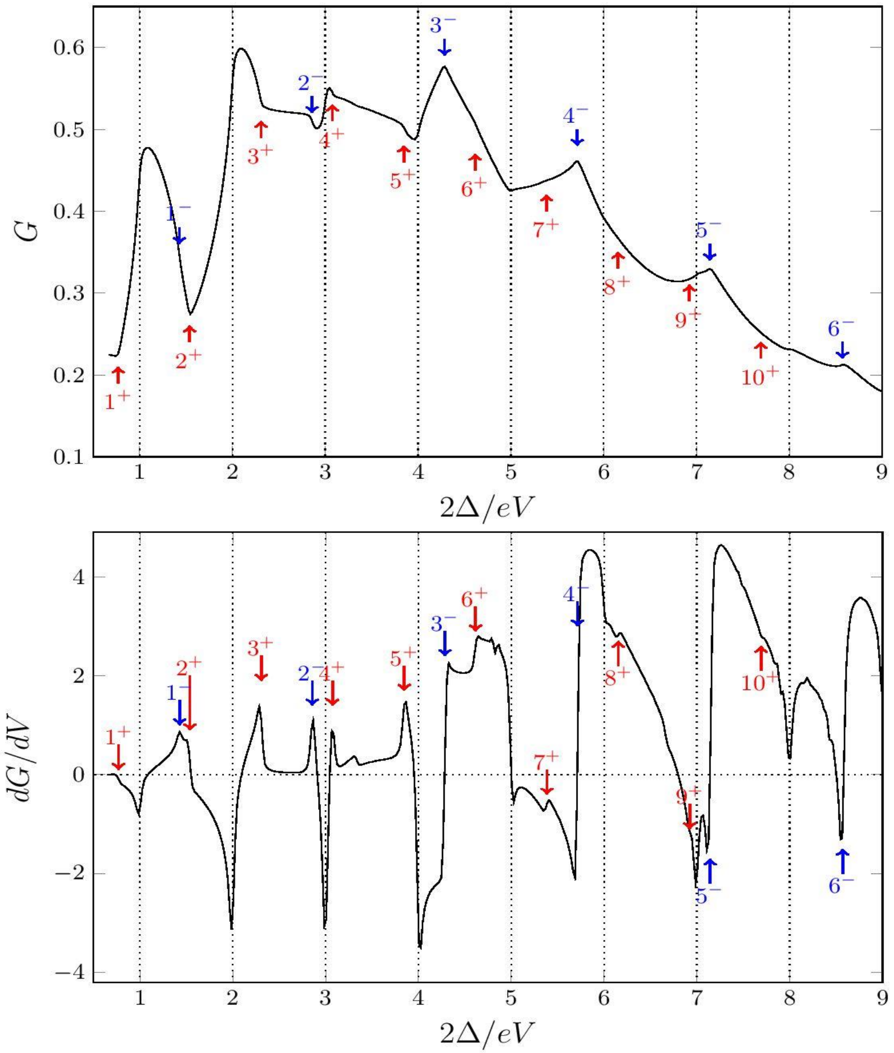
Resonant MAR features at

standard points : $\frac{2\Delta}{eV} = n$

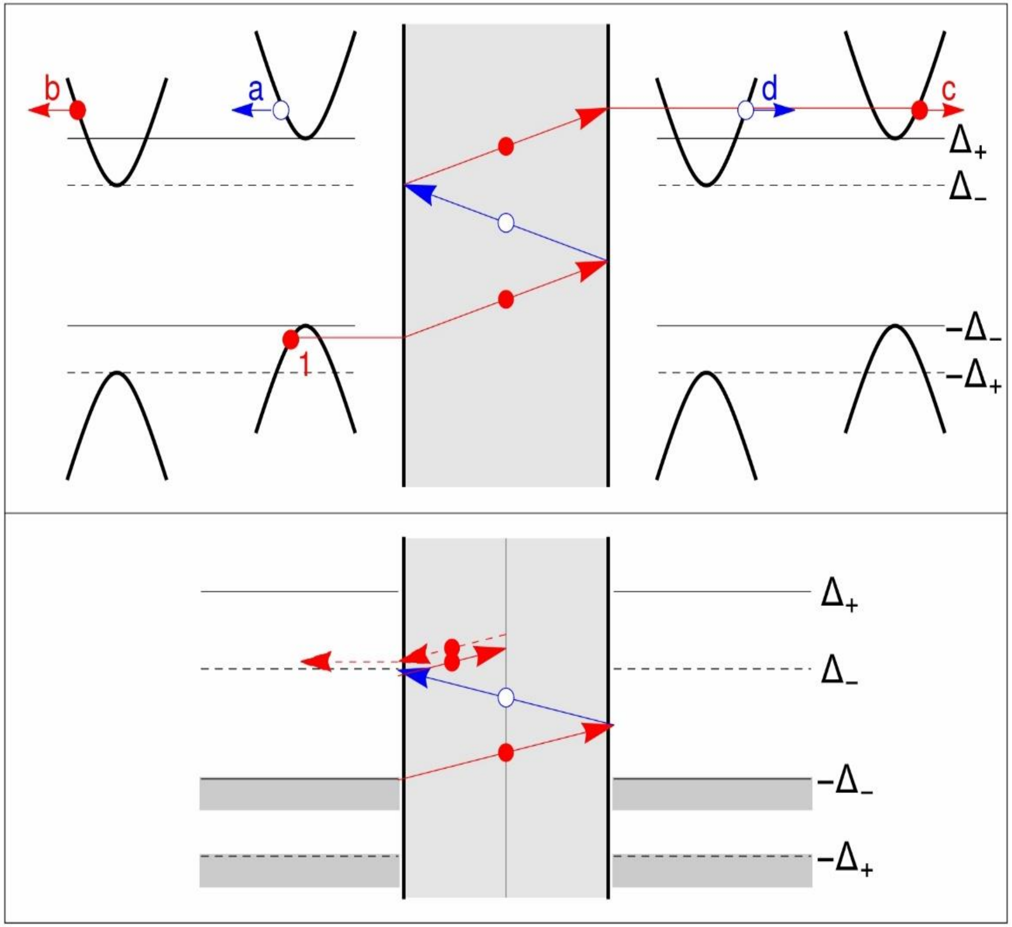
but also for Doppler-shifted

gaps (if $\tau < 1$) : $\frac{2\Delta_\pm}{eV} = n$

→ **side peaks or dips**



MAR resonances



Ballistic MAR trajectory
in energy space: resonant feature if superconducting DoS peaks align. Here for
$3eV = \Delta_+ + \Delta_- = 2\Delta$
(conventional MAR)

Normal reflection process:
MAR feature at $2eV = 2\Delta_-$
(side peak/dip from Doppler shifted gap)

Resistive rectification efficiency of Josephson diode

Resistive rectification efficiency: $\eta(V) = \frac{I(V)+I(-V)}{I(V)-I(-V)}$

Ballistic limit (T=0): $\eta(V) = \frac{4e\Delta}{hI_{q=0}(V)}|q|\xi$

$\boldsymbol{eV} \gg \boldsymbol{\Delta}$: $I_0(V) = \frac{2e^2}{h}V \quad \rightarrow \quad \eta(V) \simeq 2|q|\xi\frac{\Delta}{eV}$

$\boldsymbol{eV} \ll \boldsymbol{\Delta}$: $I_0(V) = \frac{4e\Delta}{h}\mathrm{sgn}(V) \quad \rightarrow \quad \eta(V) \simeq |q|\xi$

→ **perfect rectification ($\eta = 1$) for $|q|\xi \rightarrow 1$ & $\boldsymbol{eV} \ll \boldsymbol{\Delta}$**

→ **SDE implies highly efficient resistive rectification via MAR processes**

Numerical analysis needed for $\tau < 1$

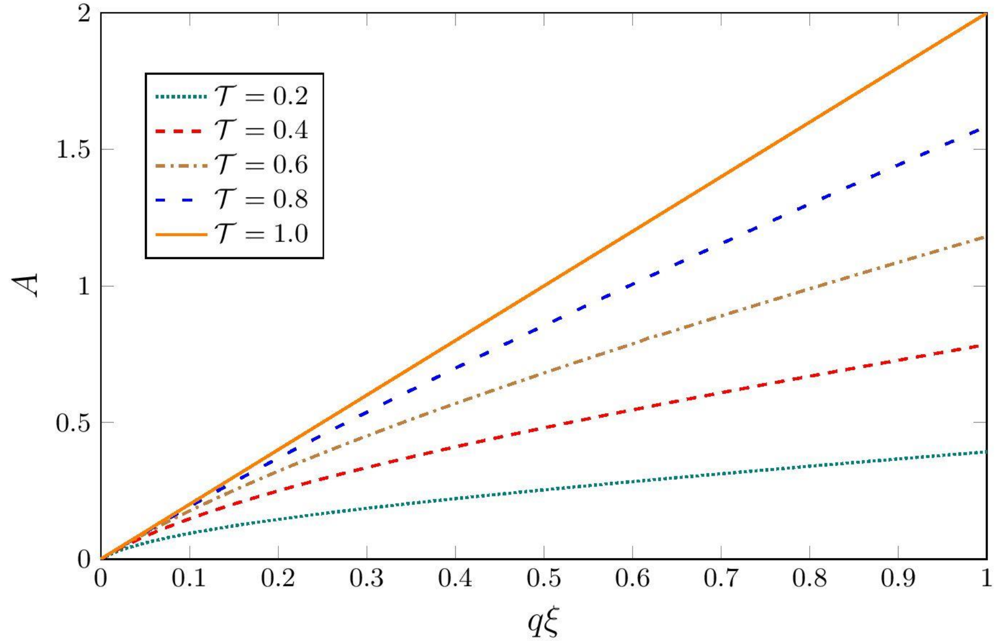
Large voltage regime

Consider regime $eV \gg \Delta$:

$$\eta(V) \simeq A(q\xi, \tau)\frac{\Delta}{eV}$$

with dimensionless prefactor A

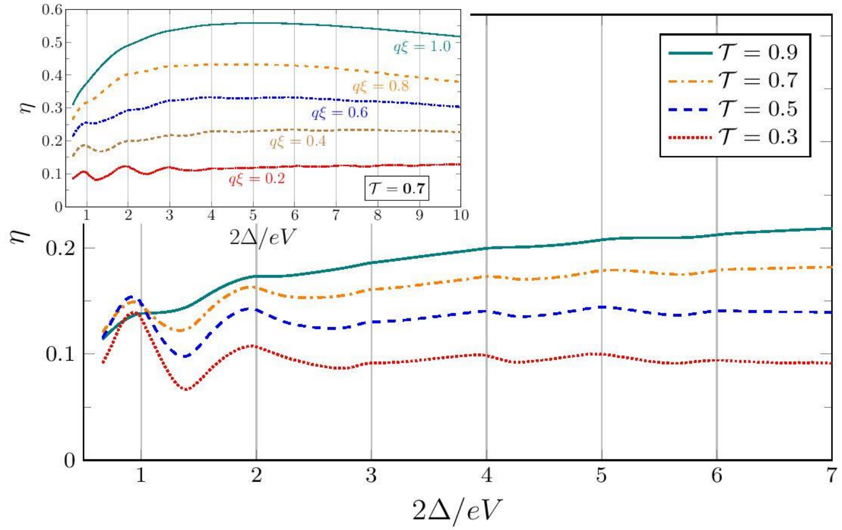
→ rectification persists even at high voltages

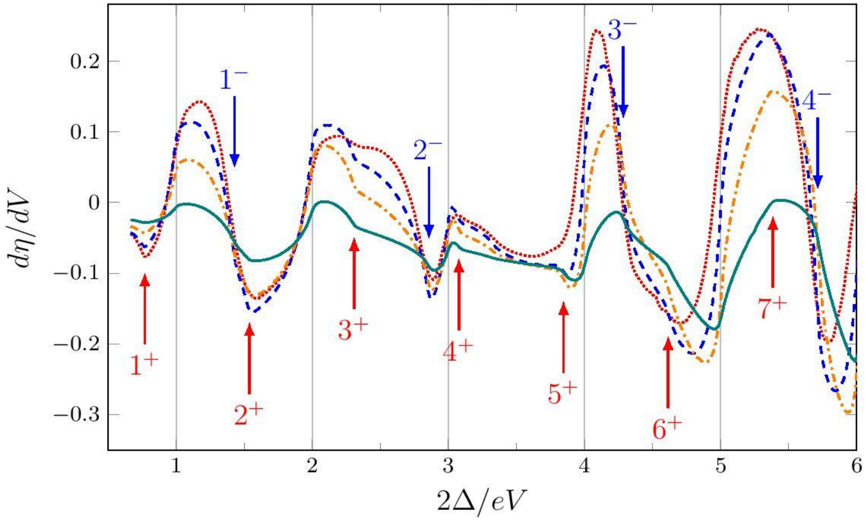


with $A(q\xi, \tau = 1) = 2|q|\xi$

Resistive rectification in Josephson diode with finite reflection

- **Finite reflection quickly degrades rectification efficiency** (as for SDE case)
- **Maximal efficiency always for $q\xi \to 1$** (unlike SDE case)
- Side features better visible in the derivative





Case study: Junction with Rashba dot

Consider 2D dot with Rashba spin-orbit coupling α and in-plane Zeeman field $\vec{b}$, start with $U = 0 \rightarrow$ exactly solvable

- N relevant orbital energy levels ε_n for $\alpha = B = \Gamma = 0$, real-valued spinor wave functions $\chi_n(\vec{r})$, tunnel contacts at $x = \pm\frac{L}{2}, y = 0$

Dot Hamiltonian:

Pauli matrices σ_a in spin space

$$H_{dot} = \sum_n d_n^+ (\varepsilon_n + \vec{b} \cdot \vec{\sigma}) d_n - i \sum_{nn'} d_n^+ (\vec{a}_{nn'} \cdot \vec{\sigma}) d_{n'}$$

$$\vec{a}_{nn'} = \frac{\alpha}{m} \int d\vec{r} \chi_n(\vec{r}) \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix} \chi_{n'}(\vec{r})$$

Rashba SOI matrix potential, similar also for Dresselhaus SOI

Tunnel Hamiltonian: $$H_{tun} = \sum_{nj\vec{k}\sigma} \left(t_{jn} c_{j\vec{k}\sigma}^+ d_{n\sigma} + h.c. \right)$$

$\rightarrow N \times N$ hybridization matrices Γ_L, Γ_R with $\Gamma_{j,nn'} = \pi \nu_0 t_{jn}^* t_{jn'}$

BCS leads: $$H_{BCS} = \sum_{j\vec{k}\sigma} \varepsilon_k c_{j\vec{k}\sigma}^+ c_{j\vec{k}\sigma} + \sum_{j\vec{k}} (\Delta e^{\mp\frac{i\varphi}{2}} c_{j\vec{k}\uparrow}^+ c_{j(-\vec{k})\downarrow}^+ + h.c.)$$

Josephson current: Exact solution

Dell'Anna, Zazunov, Egger & Martin, PRB 2007

Anomalous Josephson effect most pronounced for $\vec{b} = B\vec{e}_x$ but absent for $\vec{b} = B\vec{e}_y$ $\rightarrow$ focus on magnetic field ‖ $\vec{e}_x$

After gauging out $\vec{a}_x$ term: *for details, see Sun, Wang & Guo, PRB 2005*

$$I(\varphi) = -\frac{2e}{\hbar}\int_0^\infty d\omega\, \partial_\varphi\, tr \ln S(\omega)$$ with $4N \times 4N$ matrix

$$S(\omega) = -i\omega\left(1 + \frac{\Gamma_L+\Gamma_R}{\sqrt{\omega^2+\Delta^2}}\right) + W\sigma_z\tau_z + Z + \frac{\Delta}{\sqrt{\omega^2+\Delta^2}}Y$$

with ω-independent matrices: $W = diag(\varepsilon_n - \frac{\alpha^2}{2m})$

$$Y = (\Gamma_L + \Gamma_R)\cos\left(\frac{\varphi}{2}\right)\sigma_x\tau_z + (\Gamma_L - \Gamma_R)\sin\left(\frac{\varphi}{2}\right)\sigma_y$$

$$Z = iA_x\sigma_x\tau_x + iA_z\sigma_z - M_x\tau_y + M_z\tau_z$$

Pauli matrices τ_a in Nambu (particle-hole) space

Exact solution

with spin-orbit vector of real $N \times N$ antisymmetric matrices:

$$\vec{A}_{nn'} = \frac{\alpha}{m} \int d\vec{r}\, \chi_n(\vec{r}) \partial_y \chi_{n'}(\vec{r}) \begin{pmatrix} 2\sin^2(\alpha x) - 1 \\ 0 \\ \sin(2\alpha x) \end{pmatrix} \propto \alpha$$

and magnetic field vector of real $N \times N$ symmetric matrices:

$$\vec{M}_{nn'} = B \int d\vec{r}\, \chi_n(\vec{r}) \chi_{n'}(\vec{r}) \begin{pmatrix} 1 - 2\sin^2(\alpha\, x) \\ 0 \\ -\sin(2\alpha x) \end{pmatrix} \propto B$$

Exact expression $\rightarrow$ $I_a = 0$ for $\alpha B = 0$ $\rightarrow$ **both SOI and Zeeman field needed for anomalous Josephson effect!**

Analytical progress: assume small SOI and weak Zeeman field $\rightarrow$ $I_a \propto \alpha B$ explicitly computable

Anomalous supercurrent

Decompose $S = S_0 + S_1$ with $S_0 = S(\alpha = B = \varphi = 0)$

$$S_1 = Z + \frac{\varphi}{2}\frac{\Delta}{\sqrt{\omega^2+\Delta^2}}(\Gamma_L - \Gamma_R)\sigma_y + O(\varphi^2)$$

Perturbation series:

$$I_a = I(\varphi = 0) = \frac{2e}{\hbar}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\int_0^{\infty} d\omega\, \partial_\varphi\, tr(S_0^{-1}S_1)^n$$

first non-vanishing contribution: $n = 3$

For $\Delta \ll \max \Gamma_{d,nn}$ with $\Gamma_d = diag(\Gamma_L + \Gamma_R)$:

$$I_a = \frac{2e\Delta}{\hbar} tr_{dot}\left(\vec{M}\cdot\vec{A}\frac{1}{\Gamma_d^2+W^2}[\Gamma_L, \Gamma_R]\frac{1}{\Gamma_d^2+W^2}\right) \propto \alpha B$$

Conditions for φ_0-junction behavior

Zazunov, Egger, Jonckheere & Martin, PRL 2009

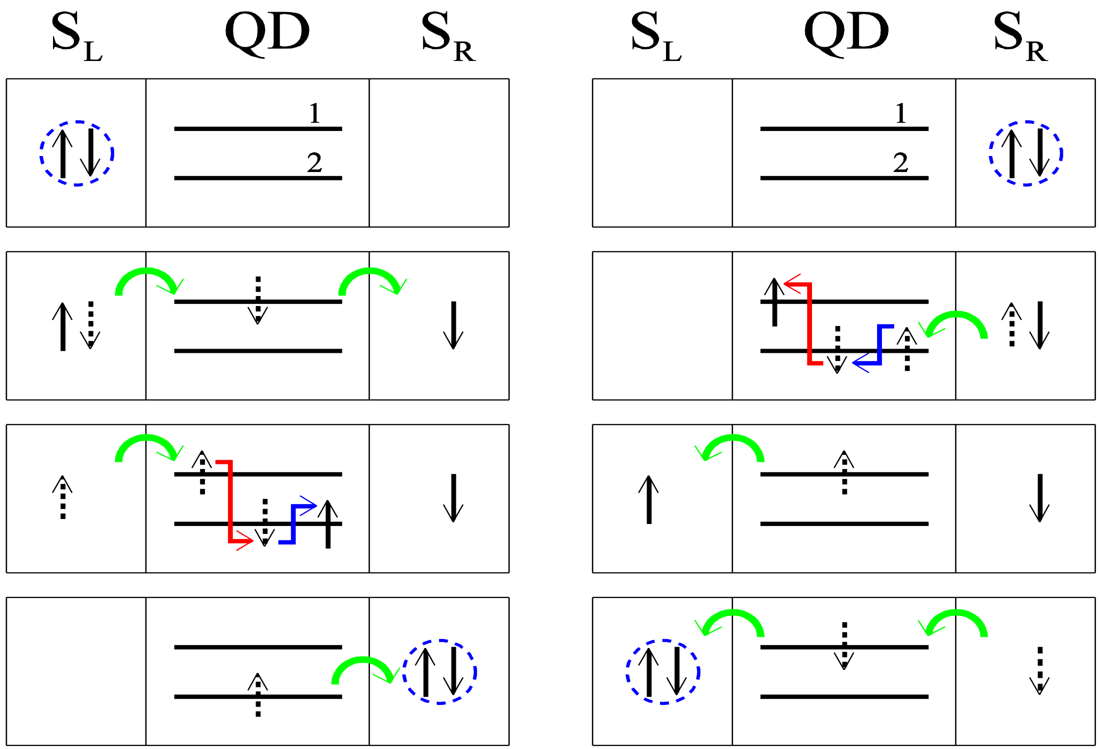
- Finite Zeeman field vector with $\vec{M} \cdot \vec{A} \neq 0$:
  broken TRS & field $\vec{b}$ not parallel to $\vec{e}_y$
- Finite spin-orbit vector:
  need $\alpha \neq 0$ & at least two dot levels ($N \geq 2$)
- Broken chirality condition: $[\Gamma_L, \Gamma_R] \neq 0$
  need broken inversion symmetry and $N \geq 2$

Summary: condition for anomalous supercurrent

$$\vec{M} \cdot \vec{A} \, [\Gamma_L, \Gamma_R] \neq 0$$

NB: condition also valid beyond perturbative regime & for $U > 0$

Basic explanation: Case $N = 2$



(a) (b)

Transfer of Cooper pair through dot for $\varphi = 0$

SOI and Zeeman field to lowest order αB give

$$H' = \sigma_x \begin{pmatrix} B & iA \\ -iA & B \end{pmatrix}$$

$$A = (A_x)_{12} \propto \alpha$$

assume real t_{jn}

Example (a) gives correction $\delta t_{L\to R} = (t_{L1} t_{R1})(t_{L1} iAB\, t_{R2})$

Example (b) yields correction $\delta t_{R\to L} = (t_{R2} B(-iA) t_{L1})(t_{R1} t_{L1})$

Constructive interference

- Both contributions to I_a are identical & add up:

$$\delta I_a^{(a)} \propto vAB\,\Gamma_{L,11}\Gamma_{R,12}$$

$$\delta I_a^{(b)} \propto (-v)(-A)B\Gamma_{L,11}\Gamma_{R,12}$$

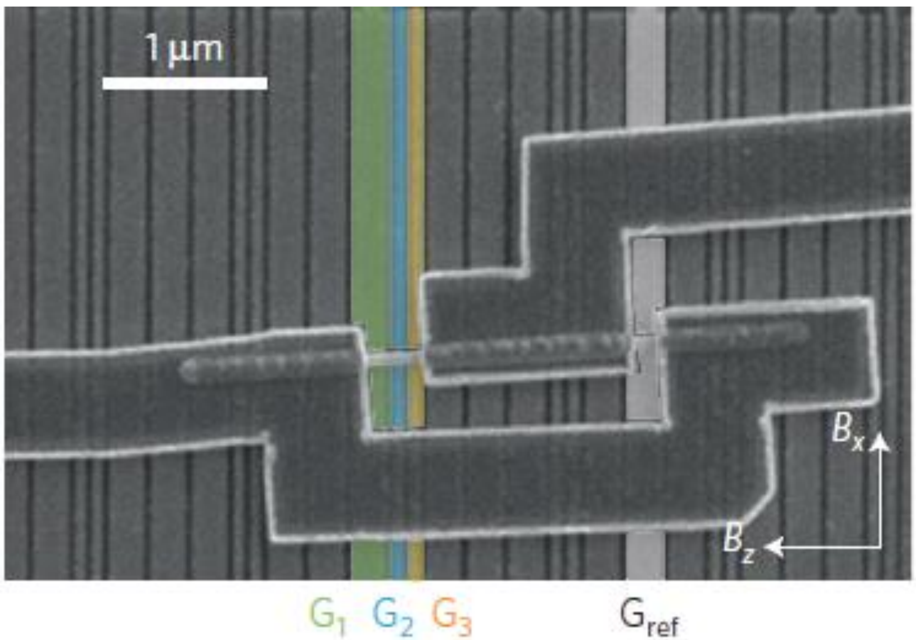
- Summing over all possible processes gives

$$I_a \propto vAB\,\left((\Gamma_{L,11}-\Gamma_{L,22})\Gamma_{R,12}-(\Gamma_{R,11}-\Gamma_{R,22})\Gamma_{L,12}\right)$$

$\rightarrow$ $I_a \neq 0$ indeed requires $\alpha B\,[\Gamma_L, \Gamma_R] \neq 0$

Experimental observation of anomalous Josephson effect

Szombati, Nadj-Perge, Car, Plissard, Bakkers & Kouwenhoven, Nature Physics 2016



- InSb nanowire dot with $N = 2$ levels and in-plane magnetic field
- SQUID geometry with NbTiN as superconductor
- Gate-tunable φ_0-shifted CPR observed

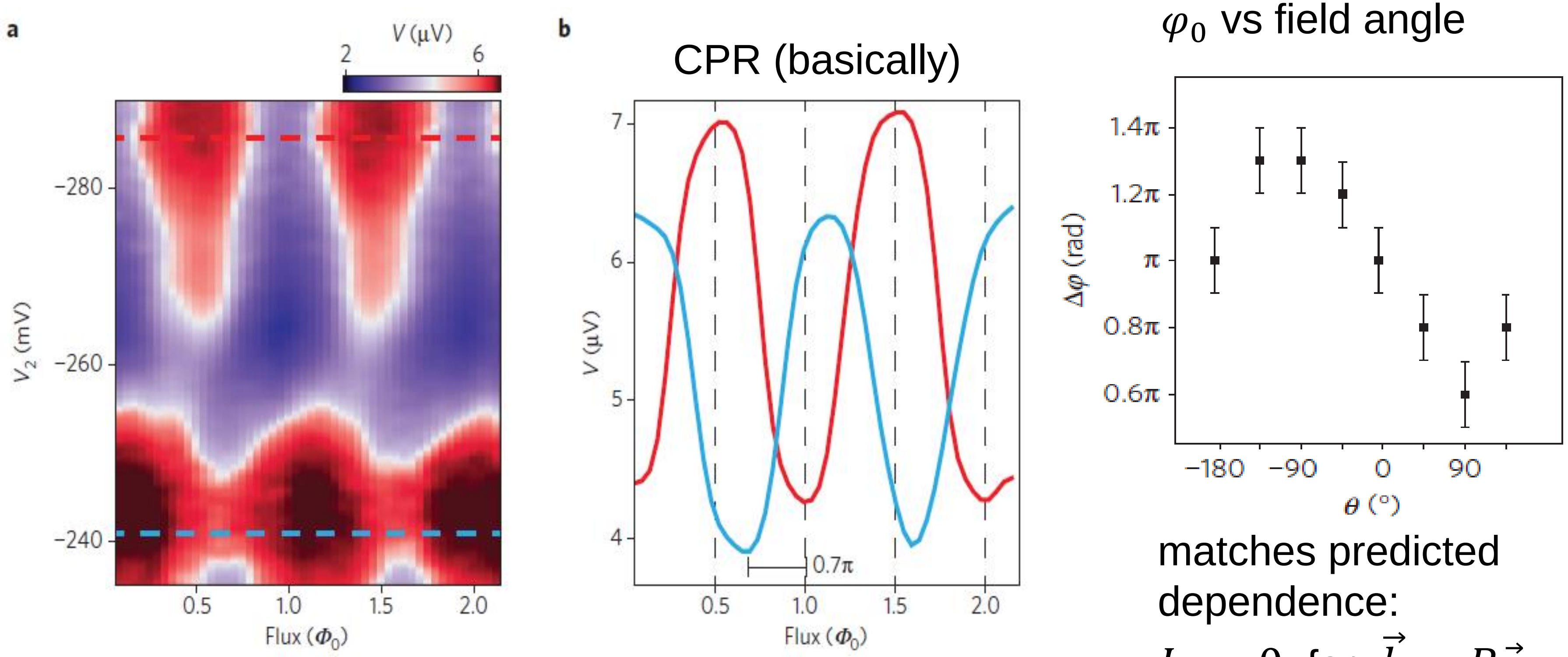
Measurement of voltage across SQUID vs flux (at constant current) → CPR

Experimental evidence

Szombati et al., Nat. Phys. 2016

CPR (basically)

φ_0 vs field angle



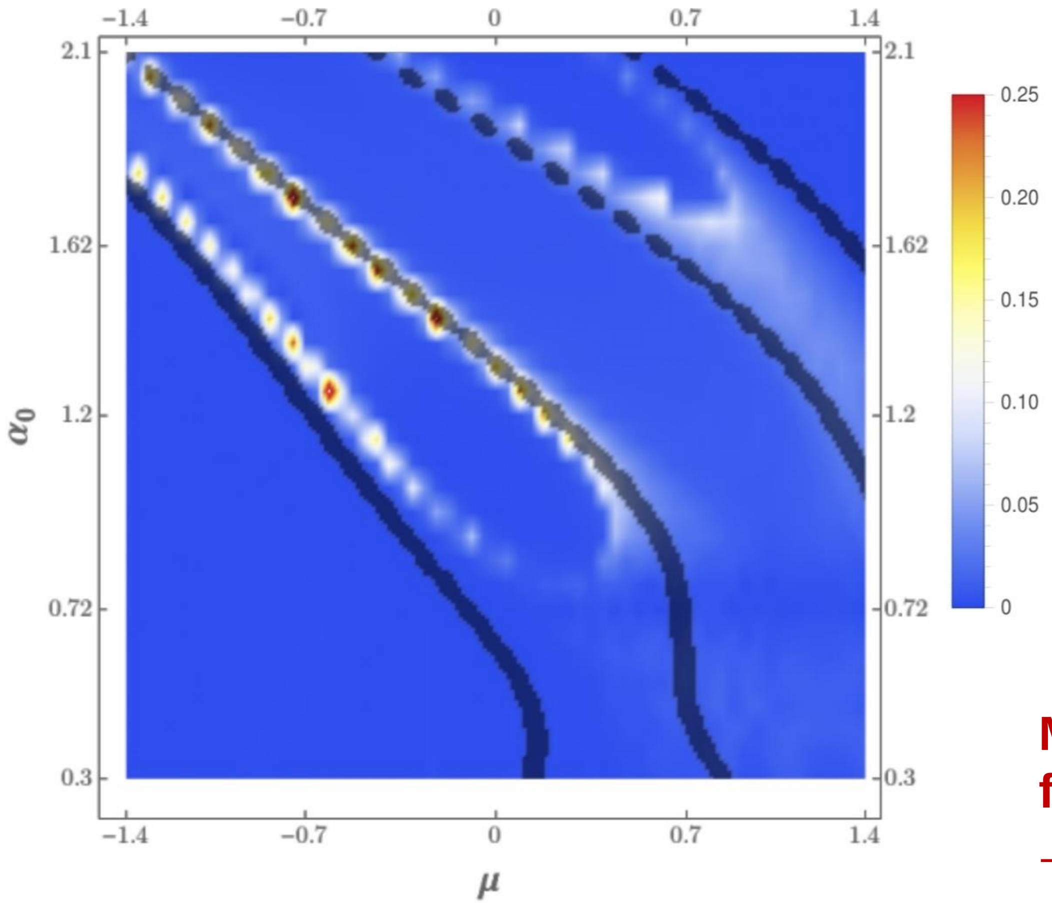
matches predicted dependence:
$I_a = 0$ for $\vec{b} = B\vec{e}_y$

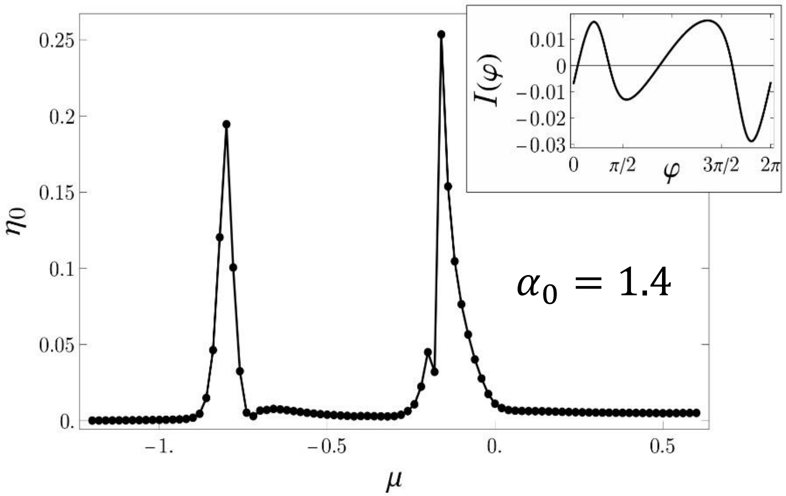
Monitor voltage drop vs gate-voltage / flux plane:
Red: $\varphi_0 = 0$ reference case

see also: Mayer, Shabani et al., Nature Comm. 2020
Strambini, Giazotto et al., Nature Nanotechn. 2021
Wesdorp et al., arXiv:2208.11198

Numerical solution for SDE efficiency

weak link = **two-level quantum dot** with Zeeman field, spin-orbit coupling α_0 & dot potential μ





Maximal SDE efficiency ≈ 25%
for resonant energy levels
→ high junction transparency

Conclusions

- **Introduction:** *Superconducting Diode Effect (SDE)*
- **Minimal model:** Josephson junction with finite Cooper pair momentum
- **Equilibrium properties:** Andreev bound states, anomalous Josephson effect & SDE
- ***Finite bias voltage rectification & Multiple Andreev reflection (MAR)***
- ***Open issues: microscopic understanding of SDE mechanisms, application potential, …***

Zazunov, Rech, Jonckheere, Grémaud, Martin & Egger, arXiv:2307.14698 & arXiv:2307.15386

THANK YOU FOR YOUR ATTENTION!